

DERIVED FUNDAMENTAL GROUPS FOR TATE MOTIVES

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ABSTRACT. We construct derived fundamental group schemes for Tate motives over connected smooth schemes over fields. We show that there exists a pro affine derived group scheme over the rationals such that its category of perfect representations models the triangulated category of rational mixed Tate motives. Under a hypothesis which is weaker than an integral version of the Beilinson-Soulé vanishing conjecture we show that there is an affine derived group scheme over the integers such that its perfect representations model Tate motives with integral coefficients. The hypothesis is for example fulfilled for number fields. This generalizes previous non-derived constructions of fundamental group schemes for Tate motives with rational coefficients.

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Date: November 2, 2010.

1991 Mathematics Subject Classification. 14F35, 14F42.

Key words and phrases. motivic fundamental groups, Tate motives, derived group schemes.

1. INTRODUCTION

The main purpose of this paper is to furnish an unconditional construction of fundamental group schemes for triangulated rational mixed Tate motives of smooth connected schemes over fields. To our knowledge this construction is the first one with the property that the fundamental group models Tate motives over such general base schemes. In order to achieve this we shall make use of a derived formalism which enables us to avoid previous assumptions such as the Beilinson-Soulé vanishing conjecture or the $K(\pi, 1)$ -conjecture.

It turns out that in general we have to consider pro-objects in the category of affine derived group schemes in order for the representation category to model the Tate motives in a correct way. We will explain this more detailed later in the introduction.

Under an integral vanishing assumption which is weaker than an integral version of the Beilinson-Soulé vanishing conjecture we show that there is an affine derived group scheme over the *integers* modelling Tate motives.

This applies in particular to Tate motives over a number field. The resulting group scheme over the integers can be thought of as a natural integral structure on the usual (non-derived) rational Tate motivic fundamental group of such a field.

Let us recall in which environment our considerations take place. Triangulated and abelian categories of mixed Tate motives have been constructed in different ways [1], [2], [10], [5], [22], [4]. One possibility is to start with a triangulated category of mixed motives $\mathrm{DM}(S)$ over the base scheme S and to consider the full triangulated subcategory generated by the Tate motives (see e.g. [16]). In general it is still unclear which construction of $\mathrm{DM}(S)$ is the correct one, i.e. for arbitrary base schemes such as $\mathrm{Spec}(\mathbf{Z})$ with general coefficients such as the integers.

In [10] Tate motives with integer coefficients over a field k have been constructed by considering module categories over Adams graded (i.e. possessing an additional grading) E_∞ -algebras in the category of chain complexes of abelian groups which come from cycle complexes with partial multiplications. It is still not settled if the resulting triangulated categories fully embed into $\mathrm{DM}(k)$ as the Tate objects. In [17] we gave another construction of Adams graded E_∞ -algebras together with an embedding of the resulting module category into $\mathrm{DM}(k)$.

We switch now to rational coefficients. In this case embeddings into $\mathrm{DM}(k)_\mathbb{Q}$ of module categories over rational cycle dga's have already been constructed in [19] and [18], see [13, II.5.5.4, Th. 111, II.5.5.5] for a summary.

In the case where the Beilinson-Soulé vanishing conjecture holds for the field k the rational triangulated category of mixed Tate motives over k admits a t -structure, see [12]. It is possible to describe the heart of this t -structure as the representations of an affine group scheme over \mathbb{Q} , see [10]. In case the $K(\pi, 1)$ -conjecture holds for the field k the triangulated category of Tate motives is then the derived category of this abelian representation category (we note that we do not distinguish between big and small categories of Tate motives in this introduction).

In this article we want to generalize this picture in two different ways: First we want to describe unconditionally the full triangulated category as representation category and second we want to get rid of rational coefficients.

For the first point it is necessary in general to pass from group schemes to so-called derived group schemes and pro derived group schemes. We shall explain these notions below. It turns out that the same procedure also settles the second issue if our weak vanishing assumption is fulfilled, see further below.

Since we work intrinsically derived it is also not necessary that a version of the $K(\pi, 1)$ -conjecture holds. Moreover our approach builds on the particular case where a vanishing result holds which is weaker than the Beilinson-Soulé vanishing conjecture. We will state this condition further below in the introduction. In this special case a description of Tate motives as representations is possible without passing to pro objects (on the side of the group schemes) and with integral coefficients.

Let us recall how the group scheme whose representation category is the heart of the above mentioned t -structure is constructed. One starts with an augmented (additionally Adams graded) commutative dga A over \mathbb{Q} modelling Tate motives. The Bar construction $B(\mathbb{Q}, A, \mathbb{Q})$, using the augmentation, gives a simplicial commutative Adams graded cdga, and the total complex inherits the structure of an Adams graded dg Hopf algebra. The zeroth cohomology of this object is the Hopf algebra representing the (non-derived) group scheme known from e.g. [10].

The algebra underlying this total object has the natural description as a derived tensor product $\mathbb{Q} \otimes_A^{\mathbf{L}} \mathbb{Q}$, and in fact it is the first stage in the Čech resolution of the augmentation $A \rightarrow \mathbb{Q}$. We refer to [23] for the geometric intuition of cosimplicial path spaces underlying this construction.

The step to produce a truly derived group scheme is to consider the full Čech resolution as a cosimplicial algebra, i.e. the object $[n] \mapsto B^n := \mathbb{Q}^{\otimes_A^{\mathbf{L}}(n+1)}$. A representation of this derived group scheme is then a cosimplicial B^\bullet -module M^\bullet which is homotopy cartesian, i.e. for any map $\varphi: [n] \rightarrow [m]$ in Δ the natural map $M^n \otimes_{B^n}^{\mathbf{L}} B^m \rightarrow M^m$ is an equivalence.

This picture has an immediate generalization to integral coefficients. Then instead of working with strictly commutative dga's one has to work with E_∞ -algebras.

Let us first briefly indicate what an affine derived group scheme is in general over an E_∞ -algebra A . It is defined to be a cosimplicial A -algebra B^\bullet which satisfies three conditions: i) the so-called Segal conditions, i.e. the natural maps $(B^1)^{\otimes_A^{\mathbf{L}} n} \rightarrow B^n$ are equivalences, ii) a condition for being group-like, iii) the map $A \rightarrow B^0$ is an equivalence.

We will construct an Adams graded affine derived group scheme in the following way: we start with an augmented Adams graded E_∞ -algebra A modelling Tate motives over some given base (with integral coefficients). This has a natural augmentation $A \rightarrow \mathbf{1}$. We view the map $\mathrm{Spec}(\mathbf{1}) \rightarrow \mathrm{Spec}(A)$ as a covering of $\mathrm{Spec}(A)$ and build the cosimplicial algebra $[n] \mapsto B^n := \mathbf{1}^{\otimes_A^{\mathbf{L}}(n+1)}$ which can be considered as the Čech resolution of this covering. The cosimplicial A -algebra B^\bullet is then an affine derived group scheme. As above we can talk about representations of this group scheme.

A representation M^\bullet of B^\bullet will be called perfect if every M^n is a perfect B^n -module, or equivalently if M^0 is a perfect B^0 -module. We denote the category of perfect representations of B^\bullet by $\mathrm{Perf}(B^\bullet)$.

We need an additional notion for A to formulate our intermediate result. For an Adams graded object X let us denote by $X(k)$ the part of X sitting in Adams degree k . Since A models Tate motives it will be of the sort that $A(k) \simeq 0$ for $k > 0$ and the unit map $\mathbf{Z} \rightarrow A(0)$ is an equivalence. We will say that A is of bounded Tate type if each complex $A(k)$, $k < 0$, is cohomologically bounded from below.

Our first main theorem states then that if A is of bounded Tate type then the category of perfect representations of B^\bullet is equivalent to the category of perfect A -modules.

This can be viewed as the statement that the map $\mathrm{Spec}(\mathbf{1}) \rightarrow \mathrm{Spec}(A)$ is really a covering in the sense that it satisfies descent for perfect modules.

These statements take place in the world of Adams graded complexes. By proposition (8.5) representations of \mathbb{G}_m are Adams graded chain complexes, so we can consider the semi-direct product of our affine derived group scheme B^\bullet with \mathbb{G}_m to obtain a group scheme whose perfect representations in chain complexes gives back the triangulated (or ∞ -) category of mixed Tate motives over the given base.

In the general case, i.e. where our weakened version of the Beilinson-Soulé vanishing conjecture is not satisfied, we have to consider a pro-system of affine derived group schemes over the rationals which arises in the following way: We write an Adams graded cdga A modelling Tate motives over the rationals as the filtered (homotopy) colimit of algebras A_i which are finitely presented. For each of the A_i we built the affine derived group scheme B_i^\bullet as above. This defines a pro affine derived group scheme “ $\lim_i B_i^\bullet$ ”. The category of perfect representations $\mathrm{Perf}(\text{“}\lim_i B_i^\bullet\text{”})$ of “ $\lim_i B_i^\bullet$ ” is defined to be the colimit of the $\mathrm{Perf}(B_i^\bullet)$. Our second main theorem states that $\mathrm{Perf}(\text{“}\lim_i B_i^\bullet\text{”})$ is equivalent to the category of geometric mixed Tate motives as tensor triangulated categories.

We remark that this relationship can be compared to the envisaged theory of derived Tannakian duality of [21]. We are able to write a given category as the category of perfect representations of a (pro) affine derived group scheme. We note that contrary to the conditions stated in loc. cit. we do not need the existence of a t -structure. We note that the statements of loc. cit. are still conjectural.

Acknowledgements. I thank Paul-Arne Østvær and the university of Oslo for the hospitality which I enjoyed during the preparation of this text. I thank Peter Arndt for helpful suggestions and discussions.

2. STATEMENT OF THE MAIN RESULTS

Let k be a field, X a smooth connected k -scheme (of finite type). Then the triangulated category $\mathrm{DM}(X)$ of motives over X is defined (see [4, Definition 10.1.1]).

We denote by $\mathrm{DMT}(X)$ the full triangulated subcategory of $\mathrm{DM}(X)$ generated by the $\mathbf{Z}(i)$, $i \in \mathbf{Z}$, and closed under sums. Also let $\mathrm{DMT}_{\mathrm{gm}}(X)$ be the full triangulated subcategory of $\mathrm{DMT}(X)$ generated by the $\mathbf{Z}(i)$.

We denote by $\mathrm{DM}(X)_R$, $\mathrm{DMT}(X)_R$ and $\mathrm{DMT}_{\mathrm{gm}}(X)_R$ the versions with R -coefficients for a commutative ring R .

Our main theorem for general bases and with rational coefficients reads as follows:

Theorem 2.1. *There is a pro affine derived group scheme “ \lim_i ” B_i^\bullet over \mathbb{Q} (for the definition of affine derived group scheme see definition (5.1) and of pro affine derived group scheme definition (5.5)) such that the category of perfect representations $\text{Perf}(\text{“}\lim_i\text{” } B_i^\bullet)$ of “ \lim_i ” B_i^\bullet (see also section (5)) is naturally equivalent to $\text{DMT}_{\text{gm}}(X)_{\mathbb{Q}}$ as tensor triangulated category. For a commutative \mathbb{Q} -algebra R let $B_{i,R}^\bullet = B_i^\bullet \otimes_{\mathbb{Q}}^{\mathbf{L}} R$. Then we have an equivalence of tensor triangulated categories $\text{DMT}_{\text{gm}}(X)_R \simeq \text{Perf}(\text{“}\lim_i\text{” } B_{i,R}^\bullet)$.*

We give the proof at the end of section (9).

We remark that this theorem has an analogue for Beilinson motives, where more general base schemes (which need not lie over a field) can be considered, see section (10.6).

Next we turn to our statements for general coefficients.

Theorem 2.2. *Suppose for each $i > 0$ there is an $N \in \mathbf{Z}$ such that*

$$\text{Hom}_{\text{DM}(X)}(\mathbf{Z}(0), \mathbf{Z}(i)[n]) = 0$$

for $n < N$. Then there is an affine derived group scheme B^\bullet over \mathbf{Z} such that the category of perfect representations $\text{Perf}(B^\bullet)$ of B^\bullet is naturally equivalent to $\text{DMT}_{\text{gm}}(X)$ as tensor triangulated category.

For a commutative ring R let $B_R^\bullet = B^\bullet \otimes^{\mathbf{L}} R$. Then we have an equivalence of tensor triangulated categories $\text{DMT}_{\text{gm}}(X)_R \simeq \text{Perf}(B_R^\bullet)$.

We give the proof at the end of section (9).

The affine derived group scheme B^\bullet can therefore be viewed as the *derived motivic fundamental group* of $\text{DMT}_{\text{gm}}(X)$ (or rather of the infinity categorical version of $\text{DMT}_{\text{gm}}(X)$).

We also have the following generalization of theorem (2.2).

Theorem 2.3. *Let R be a commutative ring of finite homological dimension. Suppose for each $i > 0$ there is an $N \in \mathbf{Z}$ such that*

$$\text{Hom}_{\text{DM}(X)_R}(R(0), R(i)[n]) = 0$$

for $n < N$. Let $R \rightarrow R'$ be a map of commutative rings. Then there is an affine derived group scheme B^\bullet over R' such that the category of perfect representations $\text{Perf}(B^\bullet)$ of B^\bullet is naturally equivalent to $\text{DMT}_{\text{gm}}(X)_{R'}$ as tensor triangulated category.

The proof is also given at the end of section (9).

We give examples of situations where these theorems apply in section (10).

3. PRELIMINARIES AND NOTATION

In this text we deal with the theory of \mathbb{S} -modules and algebras in the algebraic setting as developed in [10]. Here \mathbb{S} denotes the monoid $\mathcal{L}(1)$, where \mathcal{L} is the image of the topological linear isometries operad in $\mathbf{Cpx}(\mathbf{Ab})$ under the normalized chain complex functor. Note that in [10] the notation \mathbb{C} is used instead of \mathbb{S} . The category of \mathbb{S} -modules is endowed with the tensor product

$$M \boxtimes N = \mathcal{L}(2) \otimes_{\mathbb{S} \otimes \mathbb{S}} (M \otimes N).$$

It is shown in [10] that this indeed defines a symmetric monoidal structure on $\mathbf{Cpx}(\mathbf{Ab})$ with a so-called pseudo unit. This means that there is a unitality map $\mathbf{Z} \boxtimes M \rightarrow M$ for any M satisfying natural properties, but this map need not be an isomorphism. We refer to [20] for more background on this material, in particular for the model structures these categories are equipped with.

What will be important for us is that the category of commutative \mathbb{S} -algebras is equipped with a proper model structure, see [14]. A commutative \mathbb{S} -algebra is the same thing as an \mathcal{L} -algebra, see [10], so the theory of commutative \mathbb{S} -algebras is the same as the theory of E_∞ -algebras for the particular E_∞ -operad \mathcal{L} .

We will also speak about \mathbb{S} -modules and algebras in categories which are related to $\mathbf{Cpx}(\mathbf{Ab})$ such as $\mathbf{Cpx}(\mathbf{Ab})^{\mathbf{Z}}$ (Adams graded complexes), $\mathbf{Cpx}(R)$, the category of chain complexes of R -modules for a commutative ring R , $\mathbf{Cpx}(R)^{\mathbf{Z}}$ and complexes of modules over a cosimplicial commutative algebra.

For a commutative \mathbb{S} -algebra A we write $\mathbf{D}(A)$ for the homotopy category of A -modules. It is a closed symmetric monoidal category.

We write $\mathbf{Perf}(A)$ for the full tensor subcategory of $\mathbf{D}(A)$ of perfect objects.

We usually write $\otimes^{\mathbf{L}}$ for the derived tensor product, for example the tensor product in $\mathbf{D}(A)$ will be written $\otimes_A^{\mathbf{L}}$. If we have two \mathbb{S} -algebra morphisms $A \rightarrow B$ and $A \rightarrow C$ we denote by $B \otimes_A C$ the pushout.

We denote by Δ the simplicial category, i.e. objects are the non-empty finite ordered sets $[n] = \{0, \dots, n\}$ and morphisms are the monotone maps.

If B^\bullet is a cosimplicial commutative \mathbb{S} -algebra we write $\mathbf{D}(B^\bullet)$ for the derived category of B^\bullet -modules, where a B^\bullet -module consists of B^m -modules M^m together with maps $M^n \rightarrow M^m$ for each map $[n] \rightarrow [m]$ in Δ , linear over the corresponding map $B^n \rightarrow B^m$. This is a particular example of a section category. This material is covered in section (4).

We write $\mathbf{D}(B^\bullet)_{\text{cart}}$ for the full subcategory of cartesian objects of $\mathbf{D}(B^\bullet)$ (sometimes we also say homotopy cartesian), i.e. for those modules M^\bullet such that for any map $[n] \rightarrow [m]$ in Δ the induced map $M^n \otimes_{B^n}^{\mathbf{L}} B^m \rightarrow M^m$ is an isomorphism in $\mathbf{D}(B^m)$.

We let $\mathbf{Perf}(B^\bullet)$ be the full subcategory of $\mathbf{D}(B^\bullet)_{\text{cart}}$ consisting of B^\bullet -modules M^\bullet such that each M^n is a perfect B^n -modules, or equivalently such that M^0 is a perfect B^0 -module.

Suppose $A \rightarrow B^\bullet$ is a coaugmented cosimplicial commutative \mathbb{S} -algebra. Then we have a natural base change map $-\otimes_A^{\mathbf{L}} B^\bullet: \mathbf{D}(A) \rightarrow \mathbf{D}(B^\bullet)$. We denote its right adjoint by Tot_A .

Let $A \rightarrow B$ be a map of commutative \mathbb{S} -algebras. We associate to this map the following coaugmented cosimplicial algebra: for any $[n] \in \Delta$ we let B^n be the $[n]$ -fold coproduct of $A \rightarrow B$ in the category of commutative \mathbb{S} -algebras under B . We refer to $A \rightarrow B^\bullet$ to the coaugmented cosimplicial algebra associated to $A \rightarrow B$.

We will usually write $\mathbf{1}$ for the tensor unit in $\mathbf{Cpx}(\mathbf{Ab})^{\mathbf{Z}}$. We will write \mathbf{Z} for the complex in $\mathbf{Cpx}(\mathbf{Ab})$ with \mathbf{Z} in degree 0.

Let \mathcal{C} be a category. For an object $M \in \mathcal{C}^{\mathbf{Z}}$ and $r \in \mathbf{Z}$ we write $M(r)$ for the corresponding object of \mathcal{C} sitting in degree r .

We denote the two possible functors $\mathbf{Cpx}(\mathbf{Cpx}(\mathbf{Ab})) \rightarrow \mathbf{Cpx}(\mathbf{Ab})$ assigning to a double complex the total complex where sums resp. products are used by Total^\oplus resp. Total^Π .

For a simplicial or cosimplicial object X of an additive category we denote by UCX the unnormalized chain complex associated to X .

In the text we use that if we have a simplicial object M_\bullet in $\mathbf{Cpx}(\mathbf{Ab})$ then its homotopy colimit can be computed by $\text{Total}^\oplus(\text{UCM}_\bullet)$. Similarly, for a cosimplicial object M^\bullet in $\mathbf{Cpx}(\mathbf{Ab})$ the homotopy limit can be computed by $\text{Total}^\Pi(\text{UCM}^\bullet)$. Thus in particular Tot_A is given by such a formula.

4. DIAGRAMS OF MODEL CATEGORIES AND COFINALITY

Let I be a small category and $\omega: I \rightarrow \mathbf{ModCat}$ be a pseudo-functor. Here \mathbf{ModCat} denotes the 2-category of model categories where the morphisms are the left Quillen functors and the 2-morphisms the natural isomorphisms.

Definition 4.1. *The category of sections of the fibered category over I^{op} corresponding to ω is denoted by $\mathbf{Sect}(\omega)$. If it exists, we endow $\mathbf{Sect}(\omega)$ with the projective model structure, i.e. the model structure such that weak equivalences and fibrations are objectwise.*

In what follows we suppose that the projective model structures on the section categories exist, which is for example the case if ω takes values in cofibrantly generated model categories.

An object $x \in \mathbf{HoSect}(\omega)$ is called *cartesian* if for every map $f: i \rightarrow j$ in I the natural morphism $\mathbf{L}\omega(f)(x_i) \rightarrow x_j$ in $\mathbf{Ho}\omega(j)$ is an isomorphism. The full subcategory of $\mathbf{HoSect}(\omega)$ of cartesian objects is denoted by $\mathbf{HoSect}(\omega)_{\text{cart}}$.

Let $F: I' \rightarrow I$ be a functor and set $\omega' := \omega \circ F$. Then we have a natural pullback functor $F^*: \mathbf{Sect}(\omega) \rightarrow \mathbf{Sect}(\omega')$ which is clearly a right Quillen functor. Let F_* be its left adjoint.

For any $i \in I$ we denote by F/i the over category relative to the functor F . We let $J_i: F/i \rightarrow I'$ be the forgetful functor and ω_i the functor $\omega' \circ J_i$. We have the restriction functor $J_i^*: \mathbf{Sect}(\omega') \rightarrow \mathbf{Sect}(\omega_i)$, which is a right Quillen functor.

Lemma 4.2. *The restriction functor J_i^* is also a left Quillen functor.*

Proof. By abstract nonsense J_i^* has a right adjoint. We show that the right adjoint of J_i^* preserves fibrations and trivial fibrations. By adjunction it is enough to show that J_i^* preserves (trivial) cofibrations of the form $\text{Hom}_{I'}(i', -) \times f$ for f a (trivial) cofibration in $\omega'(i')$, where $\text{Hom}_{I'}(i', -) \times -$ denotes the left adjoint to the evaluation functor $\text{ev}_{i'}: \mathbf{Sect}(\omega') \rightarrow \omega'(i')$. But clearly

$$J_i^*(\text{Hom}_{I'}(i', -) \times f) \cong \coprod_{\alpha \in \text{Hom}_I(F(i'), i)} \text{Hom}_{F/i}((i', \alpha), -) \times f,$$

which shows the claim. □

Corollary 4.3. *For an objectwise cofibrant section $x \in \mathbf{Sect}(\omega')$ the value of $\mathbf{LF}_*(x)$ at $i \in I$ is given as the homotopy colimit of the diagram*

$$D_i: \begin{array}{ccc} F/i & \rightarrow & \omega(i) \\ (i', F(i') \xrightarrow{f} i) & \mapsto & \omega(f)(x_{i'}) \end{array} .$$

Proof. The value of $F_*(x)$ is given as the strict colimit of the above diagram. If now x is cofibrant then by lemma (4.2) the above diagram is also a cofibrant object in the diagram category $\omega(i)^{F/i}$ with the projective model structure, hence the colimit computes the homotopy colimit. \square

Remark 4.4. *This is a model category theoretic proof (under the mild assumptions that the model structures exist) of the section category version of the general infinity category statement that the left Kan extension of a functor $I' \rightarrow K$ along a functor $I' \rightarrow I$ is computed by such (homotopy) colimits.*

Recall that a category is called *contractible* if the realization of the nerve of the category is contractible.

Lemma 4.5. *Let I be a contractible category and $D: I \rightarrow \mathcal{C}$ be a diagram in a model category \mathcal{C} such that all transition maps are weak equivalences. Then for any $i \in I$ the map from $D(i)$ to the homotopy colimit of D is also a weak equivalence.*

Proof. Note that if we would know that the diagram were equivalent to a constant diagram then the lemma would directly follow from [8, Theorem 19.6.7 (1)].

By looking at mapping spaces we reduce the problem to the dual statement for simplicial sets, i.e. we have a diagram $D: I \rightarrow \mathbf{sSet}$ where the transition maps are weak equivalences and we want to prove that the map from the homotopy limit to every $D(i)$ is a weak equivalence. We apply the Bousfield-Kan spectral sequence [3, Ch. XI, 7.1]: the E_2 -term is given by the $\mathbf{R}^p \lim \pi_q D$, $p \leq q$. But for every q the diagram $\pi_q D$ is constant over the contractible category I , thus it follows from [8, Theorem 19.6.7 (2)] that $\mathbf{R}^p \lim \pi_q D = 0$ for $p > 0$ (for $q = 1$ one gives the argument using [3, Ch. XI, 7.2 (ii)]). Thus the spectral sequence degenerates and the result follows. \square

Let I, F, ω and ω' be as in the beginning of this section. Recall from [8, Definition 19.6.1] that the functor F is called *homotopy left cofinal* if for every object $i \in I$ the over category F/i is contractible.

Lemma 4.6. *Suppose that F is homotopy left cofinal. Then the functor \mathbf{LF}_* sends cartesian objects to cartesian objects and the adjoint functors \mathbf{LF}_* and \mathbf{RF}^* restrict to mutually inverse equivalences between $\mathbf{HoSect}(\omega')_{\text{cart}}$ and $\mathbf{HoSect}(\omega)_{\text{cart}}$.*

Proof. Let $x \in \mathbf{Sect}(\omega')$ and $i \in I$. Suppose all the $x_{i'}, i' \in I'$, are cofibrant. By corollary (4.3) the value of $\mathbf{LF}_*(x)$ at i is computed as the homotopy colimit over the diagram $D_i: F/i \rightarrow \omega(i)$, $(i', f: F(i') \rightarrow i) \mapsto \omega(f)(x_{i'})$. Now if x is cartesian then all transition maps of the diagram D_i are weak equivalences, and the index category F/i is contractible by assumption, hence by lemma (4.5) the maps $D_i((i', f)) \rightarrow \text{hocolim} D_i$ are weak equivalences, too. This shows the first statement.

The unit $\text{id} \rightarrow \mathbf{R}F^* \circ \mathbf{L}F_*$ and the counit $\mathbf{L}F_* \circ \mathbf{R}F^* \rightarrow \text{id}$ are isomorphisms on cartesian objects by the same argument as for the first part. This completes the proof. \square

Lemma 4.7. *Let D be a small category and $F: \Delta \rightarrow D$ a functor. Then for any $d \in D$ the over category F/d has the homotopy type of the simplicial set $[n] \mapsto \text{Hom}_D(F([n]), d)$.*

Proof. This is as in the proof of [6, Proposition 6.11]. \square

We will apply lemma (4.6) to the functors $\text{diag}: \Delta \rightarrow \Delta \times \Delta$ and $i: \Delta \rightarrow \Delta_*$. We introduce the latter category.

First we denote by Δ_+ the category of finite ordered sets $[p]$ for $p \in \{-1\} \cup \mathbf{N}$, so that $[-1]$ is the empty ordered set. We have a natural full embedding $\Delta \hookrightarrow \Delta_+$.

Next for any $[p] \in \Delta_+$ we denote by $[p]_*$ the ordered set $[p] \sqcup \{*\}$, where we declare the ordering by $p < *$. We view $[p]_*$ as a *pointed* finite ordered set pointed by $*$. We let Δ_* be the category of the pointed finite ordered sets $[p]_*$, $[p] \in \Delta_+$, with order preserving pointed maps.

So we have an embedding of categories $\Delta \subset \Delta_*$ which is not full.

Lemma 4.8. *For any $[q]_* \in \Delta_*$ the simplicial set $[p] \mapsto \text{Hom}_{\Delta_*}([p]_*, [q]_*)$ is contractible.*

Proof. In fact any augmented simplicial set $K: \Delta_+^{\text{op}} \rightarrow \mathbf{Set}$ which can be extended to a functor $\Delta_* \rightarrow \mathbf{Set}$ has the discrete homotopy type of $A := K([-1])$. Such an extension \tilde{K} gives the following data: We denote by $\text{cs}A$ the constant simplicial set on A . The augmentation of K is a map $\epsilon: K \rightarrow \text{cs}A$. The projections $[p]_* \rightarrow [-1]_*$ define a section s of ϵ . Furthermore we get a simplicial homotopy $s \circ \epsilon \rightarrow \text{id}$ as follows: A map $K \times \Delta[1] \rightarrow K$ is a family of maps $h_p^\alpha: K_p \rightarrow K_p$, $\alpha \in \Delta[1]_p = \text{Hom}_\Delta([p], [1])$. Now for any such α we denote by r_α the map $[p]_* \rightarrow [p]_*$ which is the identity on $\alpha^{-1}(0)$ and sends $\alpha^{-1}(1)$ to $*$. Define h_p^α to be the map $\tilde{K}(r_\alpha)$. Then it is easily checked that the h_p^α fit together to a homotopy from $s \circ \epsilon$ to id . \square

Corollary 4.9. *The inclusion functors $\text{diag}: \Delta \rightarrow \Delta \times \Delta$ and $i: \Delta \rightarrow \Delta_*$ are homotopy left cofinal.*

Proof. Let $([n], [m]) \in \Delta \times \Delta$. By lemma (4.7) the over category $\text{diag}/([n], [m])$ has the homotopy type of $\Delta[n] \times \Delta[m]$, which is contractible. By the same lemma the over category $i/[q]_*$ has the homotopy type of the simplicial set

$$[p] \mapsto \text{Hom}_{\Delta_*}([p]_*, [q]_*),$$

which is contractible by lemma (4.8). \square

Proposition 4.10. *Let $F: \Delta_* \rightarrow \mathbf{ModCat}$ be a diagram of model categories (i.e. a pseudo-functor), let $F': \Delta \rightarrow \mathbf{ModCat}$ be the restricted diagram. Then the canonical functor $\text{Ho}F([-1]_*) \rightarrow \text{HoSect}(F')_{\text{cart}}$ is an equivalence.*

Proof. Consider the following 2-commutative diagram:

$$\begin{array}{ccc} \mathrm{HoSect}(F')_{\mathrm{cart}} & \longrightarrow & \mathrm{HoSect}(F)_{\mathrm{cart}} \\ \uparrow & \nearrow & \\ \mathrm{Ho}F([-1]_*) & & \end{array}$$

The inclusion $\{[-1]_*\} \rightarrow \Delta_*$ is homotopy left cofinal, thus by lemma (4.6) the diagonal arrow is an equivalence. By corollary (4.9) the inclusion $\Delta \rightarrow \Delta_*$ is homotopy left cofinal thus by lemma (4.6) the horizontal arrow is an equivalence. It follows that the vertical map is also an equivalence which was to be shown. \square

Corollary 4.11. *Let $F: \Delta \times \Delta \rightarrow \mathrm{ModCat}$ be a diagram of model categories, let $F': \Delta \rightarrow \mathrm{ModCat}$ be the composition of F with the diagonal $\Delta \rightarrow \Delta \times \Delta$. Then the canonical functor $\mathrm{HoSect}(F)_{\mathrm{cart}} \rightarrow \mathrm{HoSect}(F')_{\mathrm{cart}}$ is an equivalence.*

Proof. This follows from lemma (4.6) and corollary (4.9). \square

5. AFFINE DERIVED GROUP SCHEMES

The definitions in this section go back to [21].

Let R be a commutative \mathbb{S} -algebra.

We let $\alpha_i: [1] \rightarrow [n]$, $i = 0, \dots, n-1$, be the map $0 \mapsto i$, $1 \mapsto i+1$. We let $s: [0] \rightarrow [1]$ be the map $0 \mapsto 0$, $t: [0] \rightarrow [1]$, $0 \mapsto 1$. Finally, let $c: [1] \rightarrow [2]$ be the map $0 \mapsto 0$, $1 \mapsto 2$.

Definition 5.1. *An affine derived groupoid over R is a cosimplicial commutative R - \mathbb{S} -algebra A^\bullet such that*

- (1) *the canonical map*

$$A^1 \otimes_{t, A^0, s}^{\mathbf{L}} A^1 \otimes_{t, A^0, s}^{\mathbf{L}} \dots \otimes_{t, A^0, s}^{\mathbf{L}} A^1 \rightarrow A^n$$

(n tensor factors on the left hand side) induced by $\alpha_0, \dots, \alpha_{n-1}$ is an equivalence,

- (2) *the map*

$$A^1 \otimes_{s, A^0, s}^{\mathbf{L}} A^1 \rightarrow A^2$$

induced by c, α_0 is an equivalence.

An affine derived group scheme over R is a derived affine groupoid over R such that the map $R \rightarrow A^0$ is an equivalence.

Note that the left hand sides in (1) and (2) for an affine derived group can be written as an absolute tensor product of A^1 's.

Proposition 5.2. *Let $A \rightarrow B$ be a cofibration of commutative \mathbb{S} -algebras. Let $A \rightarrow B^\bullet$ be the corresponding coaugmented cosimplicial commutative \mathbb{S} -algebra. Then B^\bullet is an affine derived groupoid over A or equivalently over the initial algebra.*

Proof. The first property follows from the equivalence

$$(B \otimes_A B) \otimes_{t,B,s}^{\mathbf{L}} (B \otimes_A B) \otimes_{t,B,s}^{\mathbf{L}} \cdots \otimes_{t,B,s}^{\mathbf{L}} (B \otimes_A B) \xrightarrow{\sim} B \otimes_A B \otimes_A \cdots \otimes_A B = B^n,$$

where in the first entry n tensor factors are used, and in the second entry $n + 1$. The second property follows from the fact that the map

$$(B \otimes_A B) \otimes_{s,B,s}^{\mathbf{L}} (B \otimes_A B) \rightarrow B \otimes_A B \otimes_A B = B^2$$

induced by the maps c, α_0 is an equivalence. \square

Proposition 5.3. *Let $R \rightarrow A$ be a map of commutative \mathbb{S} -algebras and $A \rightarrow B$ a cofibration of commutative \mathbb{S} -algebras. Let $A \rightarrow B^\bullet$ be the corresponding coaugmented cosimplicial commutative \mathbb{S} -algebra. Suppose the map $R \rightarrow B$ is an equivalence. Then B^\bullet is an affine derived group over R .*

Proof. By proposition (5.2) B^\bullet is an affine derived groupoid. Since $R \rightarrow B = B^0$ is an equivalence the result follows. \square

Remark 5.4. *Let $R \rightarrow A$ be a cofibration of commutative \mathbb{S} -algebras. Factor the map $A' := A \otimes_R A \rightarrow A$ into a cofibration $A' \rightarrow B$ followed by a weak equivalence $B \rightarrow A$. Let $A' \rightarrow B^\bullet$ be the coaugmented cosimplicial algebra associated to $A' \rightarrow B$. Then B^\bullet is an affine derived group scheme over A , where we consider B^\bullet as lying over A via any of the two canonical morphisms $A \rightarrow A'$. This can be considered as giving the loop stack $\mathrm{Spec}(B^1) = \mathrm{Spec}(S^1 \otimes_R^{\mathbf{L}} A)$ the structure of a derived group scheme over $\mathrm{Spec}(A)$. If now $A \rightarrow C$ is a cofibration such that $R \rightarrow C$ is an equivalence then the pushforward of the loop stack with respect to the map $A \rightarrow C$ gives the construction of the affine derived group scheme given by proposition (5.3) applied to the datum $R \rightarrow A \rightarrow C$.*

Let A^\bullet be an affine derived group scheme over some commutative \mathbb{S} -algebra R . By a *representation* of A^\bullet we understand a homotopy cartesian A^\bullet -module M^\bullet . The tensor triangulated category of representations of A^\bullet is $\mathrm{D}(A^\bullet)_{\mathrm{cart}}$. A representation is called *perfect* if it is an object of $\mathrm{Perf}(A^\bullet)$.

Definition 5.5. *A pro affine derived group scheme over R is a functor $B^\bullet: i \mapsto B_i^\bullet$ from a filtered category I to the category of cosimplicial commutative R - \mathbb{S} -algebras such that for each $i \in I$ the cosimplicial algebra B_i^\bullet is an affine derived group scheme over R .*

We will write “ \lim_i ” B_i^\bullet for a pro affine derived group scheme.

The category of representations of a pro affine derived group scheme “ \lim_i ” B_i^\bullet is defined to be the 2-colimit of the representation categories of the individual affine derived group schemes B_i^\bullet , i.e. it is $2\text{-colim}_i \mathrm{D}(B_i^\bullet)_{\mathrm{cart}}$. This is a tensor triangulated category. The category $\mathrm{Perf}(\text{“}\lim_i\text{” } B_i^\bullet)$ is defined to be $2\text{-colim}_i \mathrm{Perf}(B_i^\bullet)$. It is again a tensor triangulated category.

6. TATE ALGEBRAS AND DESCENT

Let A be a commutative \mathbb{S} -algebra in $\mathbf{Cpx}(\mathbf{Ab})^{\mathbb{Z}}$. We say that A is of *Tate-type* if

- (1) $A(k) \simeq 0$ for $k > 0$,

(2) the map $\mathbf{Z} \rightarrow A(0)$ is a quasi-isomorphism.

We further say that A is of *bounded* Tate-type if $A(k)$ is cohomologically bounded from below for each $k < 0$. We say that A is of *strict* Tate-type if $A(k) = 0$ for $k > 0$ and the map $\mathbf{Z} \rightarrow A(0)$ is an isomorphism.

It is clear that for any algebra A of Tate-type there is a canonical algebra A' of strict Tate-type together with a quasi-isomorphism $A' \rightarrow A$.

Let A be an algebra of strict Tate-type. Then there is a canonical augmentation $A \rightarrow \mathbf{1}$ being the identity in Adams degree 0.

We associate to this situation an affine derived group scheme in the following way: factor the map $A \rightarrow \mathbf{1}$ into a cofibration $A \rightarrow B$ followed by a weak equivalence $B \rightarrow \mathbf{1}$. Let $A \rightarrow B^\bullet$ be the coaugmented cosimplicial algebra associated to $A \rightarrow B$. Then by proposition (5.3) B^\bullet is a derived affine group scheme over $\mathbf{1}$.

Notice that for any algebra A of Tate-type there is a canonical derived pushforward $U: D(A) \rightarrow D(\mathbf{1}) \simeq D(\mathbf{Ab})^{\mathbf{Z}}$ by first restricting to the quasi isomorphic algebra of strict Tate-type $D(A) \rightarrow D(A')$ and then taking the derived pushforward along the augmentation.

Let A be an algebra of Tate-type. We denote by $D(A)_{\text{Aba}} \subset D(A)$ the full tensor triangulated subcategory consisting of modules M such that $M(r) \simeq 0$ for r big enough. We call these modules *Adams bounded from above*.

Lemma 6.1. *Let A be an algebra of Tate-type. Then the push forward*

$$U: D(A)_{\text{Aba}} \rightarrow D(\mathbf{1}) \simeq D(\mathbf{Ab})^{\mathbf{Z}}$$

is conservative, i.e. sends non-zero objects to non-zero objects.

Proof. For the argument we replace the \mathbb{S} -algebra A by an equivalent associative algebra, also denoted by A , since then it is easier to compute the derived push forward. This can be done in the way that $A(r) = 0$ for $r > 0$ and $A(0) = \mathbf{Z}$. The algebra A has then the canonical augmentation $e: A \rightarrow \mathbf{Z}$. We denote now by $\text{Mod}(A)$ the category of left A -modules.

Now let $M \in \text{Mod}(A)$ be a module which is Adams bounded from above and non-equivalent to zero. Let r_0 be the biggest integer such that $M(r_0)$ is non-equivalent to 0. Assume M is cofibrant for the projective model structure on $\text{Mod}(A)$, so we can compute $U(M)$ by $\mathbf{1} \otimes_A M =: M'$. Furthermore we can assume $M(r) = 0$ for $r > r_0$ (use the model structure on $\{M \in \text{Mod}(A) \mid M(r) = 0, r > r_0\}$ and observe that a cofibrant object for this model structure is also cofibrant for the projective model structure on $\text{Mod}(A)$). Then clearly $M'(r) = 0$ for $r > r_0$ and $M'(r_0) = M(r_0)$, hence M' is non-equivalent to 0, too. \square

Lemma 6.2. *Let $f: A \rightarrow B$ be a cofibration of commutative \mathbb{S} -algebras and suppose f has a cosection s . Let $A \rightarrow B^\bullet$ be the corresponding coaugmented cosimplicial algebra. Then the whole category of modules over A satisfies descent with respect to the map f , i.e. the functor*

$$D(A) \rightarrow D(B^\bullet)_{\text{cart}}$$

is an equivalence.

Proof. Define a map $(I \setminus \{*\}) \otimes_A B \rightarrow (J \setminus \{*\}) \otimes_A B$ for a map between pointed finite sets $\varphi: I \rightarrow J$ by sending the summands in $(I \setminus \{*\}) \otimes_A B$ corresponding to preimages of $*$ under φ to the target by first projecting down to A via s and then using the structure map. On the remaining summands the map is defined by application of φ . By this procedure one gets a functor $\Phi: \Delta_* \rightarrow \text{Alg}(\mathcal{C})$, $[p]_* \rightarrow [p] \otimes_A B$, extending the coaugmented algebra $A \rightarrow B^\bullet$.

By proposition (4.10) it follows that the functor $D(A) \rightarrow D(B^\bullet)_{\text{cart}}$ is an equivalence, which was to be shown. \square

Lemma 6.3. *Let $f: A \rightarrow B$ be a cofibration of commutative \mathbb{S} -algebras and $A \rightarrow B^\bullet$ be the corresponding coaugmented cosimplicial algebra (i.e. $B^n = B \otimes_A \cdots \otimes_A B$ ($n+1$ times)). Let $B \rightarrow B'^\bullet$ be the pushforward of $A \rightarrow B^\bullet$ along f . Let $\mathcal{C} \subset D(A)$ be a full triangulated subcategory such that the following properties are fulfilled:*

- (1) *The functor $D(B^\bullet) \rightarrow D(B'^\bullet)$ preserves total objects for modules in $D(B^\bullet)$ which are in the image of the composition $\mathcal{C} \rightarrow D(A) \rightarrow D(B^\bullet)$, in the sense that for $M \in \mathcal{C}$, the natural map*

$$\text{Tot}_A(M \otimes_A^{\mathbf{L}} B^\bullet) \otimes_A^{\mathbf{L}} B \rightarrow \text{Tot}_B(M \otimes_A^{\mathbf{L}} B^\bullet \otimes_A^{\mathbf{L}} B)$$

is an isomorphism.

- (2) *The functor $\text{Tot}_A \circ (- \otimes_A^{\mathbf{L}} B^\bullet)$ maps \mathcal{C} to itself.*
- (3) *The composition $\mathcal{C} \rightarrow D(A) \rightarrow D(B)$ is conservative.*

Then f satisfies descent for modules in \mathcal{C} in the sense that the unit for the adjunction between $D(A)$ and $D(B^\bullet)$ is an isomorphism on objects from \mathcal{C} .

Proof. By properties (2) and (3) we can test the unit to be an isomorphism on an object $M \in \mathcal{C}$ after applying $\mathbf{L}f_*$. By property (1) this amounts to saying that $\mathbf{L}f_* M$ satisfies descent with respect to the coaugmented cosimplicial algebra $B \rightarrow B'^\bullet$. This is clearly fulfilled by Lemma 6.2. \square

Lemma 6.4. *Let $f: A \rightarrow B$, $A \rightarrow B^\bullet$ and $B \rightarrow B'^\bullet$ be as in Lemma 6.3. Let $\mathcal{D} \subset D(B^\bullet)_{\text{cart}}$ be a full triangulated subcategory such that the following properties are fulfilled:*

- (1) *The functor $D(B^\bullet) \rightarrow D(B'^\bullet)$ preserves total objects for modules in \mathcal{D} in the sense that for $M \in \mathcal{D}$, the natural map*

$$\text{Tot}_A(M) \otimes_A^{\mathbf{L}} B \rightarrow \text{Tot}_B(M \otimes_A^{\mathbf{L}} B)$$

is an isomorphism.

- (2) *The functor $(- \otimes_A^{\mathbf{L}} B^\bullet) \circ \text{Tot}_A$ maps \mathcal{D} to itself.*

Then f satisfies descent for modules in \mathcal{D} in the sense that the counit for the adjunction between $D(A)$ and $D(B^\bullet)$ is an isomorphism on objects from \mathcal{D} .

Proof. Note first that the functor $D(B^\bullet)_{\text{cart}} \rightarrow D(B'^\bullet)_{\text{cart}}$ is conservative since the map $B \rightarrow B' = B \otimes_A B$ has a retraction. Thus by property (3) we can test if the counit is an isomorphism on an object $M \in \mathcal{D}$ after applying $- \otimes_A^{\mathbf{L}} B = - \otimes_B^{\mathbf{L}} B'^\bullet$. By property (1) this amounts to saying that $M \otimes_A^{\mathbf{L}} B$ satisfies descent with respect to the coaugmented cosimplicial algebra $B \rightarrow B'^\bullet$, which is again fulfilled by Lemma 6.2. \square

Lemma 6.5. *Let A_\bullet be a simplicial commutative \mathbb{S} -algebra in $\mathbf{Cpx}(\mathbf{Ab})^{\mathbb{Z}}$ and $|A|_{\text{comm}}$ be its realization in commutative \mathbb{S} -algebras which is equivalent to the homotopy colimit of A_\bullet . Let A_\bullet^\sharp be the underlying simplicial module and $|A|_{\text{comm}}^\sharp$ the underlying module of $|A|_{\text{comm}}$. Then the natural map*

$$|A_\bullet^\sharp| \rightarrow |A|_{\text{comm}}^\sharp$$

is an isomorphism in $\mathbf{D}(\mathbf{Ab})^{\mathbb{Z}}$.

Proof. This follows from [7, VII., Prop. 3.3]. \square

Lemma 6.6. *Let A be an algebra of strict Tate-type and $e: A \rightarrow \mathbf{1}$ the corresponding augmentation. Let $A \rightarrow B \rightarrow \mathbf{1}$ be a factorization of e into a cofibration followed by a weak equivalence. Let B^\bullet be the cosimplicial algebra associated to $A \rightarrow B$. Then the homotopy type of the algebra B^1 is the same as the realization of the simplicial algebra $(A \otimes^{\mathbf{L}} \Delta^1) \otimes_{A \otimes^{\mathbf{L}} \partial \Delta^1}^{\mathbf{L}} (\mathbf{1} \otimes^{\mathbf{L}} \partial \Delta^1)$, where the map $A \otimes^{\mathbf{L}} \partial \Delta^1 \rightarrow \mathbf{1} \otimes^{\mathbf{L}} \partial \Delta^1$ is $e \otimes^{\mathbf{L}} e$.*

Proof. We first note that the realization of the simplicial algebra $A \otimes^{\mathbf{L}} \Delta^1$ is A . The simplicial algebra $A \otimes^{\mathbf{L}} \Delta^1$ is a simplicial object in $A \otimes^{\mathbf{L}} \partial \Delta^1 \simeq A \otimes^{\mathbf{L}} A$ -algebras. Realization of such algebras commutes with pushforward along any map $A \otimes^{\mathbf{L}} A \rightarrow C$ of algebras. Thus we see that the realization of $(A \otimes^{\mathbf{L}} \Delta^1) \otimes_{A \otimes^{\mathbf{L}} \partial \Delta^1}^{\mathbf{L}} (\mathbf{1} \otimes^{\mathbf{L}} \partial \Delta^1)$ is equivalent to $A \otimes_{A \otimes^{\mathbf{L}} A}^{\mathbf{L}} (\mathbf{1} \otimes^{\mathbf{L}} \mathbf{1})$ which in turn is equivalent to $\mathbf{1} \otimes_A^{\mathbf{L}} \mathbf{1}$. \square

Corollary 6.7. *Let A be an algebra of strict Tate-type and $e: A \rightarrow \mathbf{1}$ the corresponding augmentation. Let $A \rightarrow B \rightarrow \mathbf{1}$ be a factorization of e into a cofibration followed by a weak equivalence. Let B^\bullet be the cosimplicial algebra associated to $A \rightarrow B$. Then the homotopy type of the underlying module of B^1 is the same as the realization of the simplicial module $[n] \mapsto A^{\otimes^{\mathbf{L}} \Delta^1(n)} \otimes_{A \otimes^{\mathbf{L}} A}^{\mathbf{L}} (\mathbf{1} \otimes^{\mathbf{L}} \mathbf{1})$.*

Proof. This follows from Lemma 6.6 with Lemma 6.5. \square

Lemma 6.8. *Let A be an algebra of strict Tate-type and $e: A \rightarrow \mathbf{1}$ the corresponding augmentation. Let A' be a replacement of A as a strictly associative algebra of strict Tate-type such that for any k the complex $A'(k)$ is cofibrant (which always exists by using e.g. a model structure on augmented strictly associative algebras). Set $\overline{A} := A'/\mathbf{1}$. Then the realization of the simplicial module $[n] \mapsto A^{\otimes^{\mathbf{L}} \Delta^1(n)} \otimes_{A \otimes^{\mathbf{L}} A}^{\mathbf{L}} (\mathbf{1} \otimes^{\mathbf{L}} \mathbf{1})$ is computed as the total complex of a bicomplex of the form*

$$\mathbf{1} \leftarrow \overline{A} \leftarrow \overline{A}^{\otimes 2} \leftarrow \overline{A}^{\otimes 3} \leftarrow \cdots,$$

where in the last line the bare tensor product in $\mathbf{Cpx}(\mathbf{Ab})^{\mathbb{Z}}$ is used.

Proof. In fact applying degreewise the normalized associated chain complex to the two-sided simplicial Bar-complex $B_*(\mathbf{1}, A', \mathbf{1})$ which is defined analogously to [7, IV., Definition 7.2] yields exactly a bicomplex of the type indicated. Comparing the two-sided Bar-complex for associative \mathbb{S} -algebras and strictly associative algebras yields an equivalence between the realization of $B_*(\mathbf{1}, A', \mathbf{1})$ and that of the simplicial module $[n] \mapsto A^{\otimes^{\mathbf{L}} \Delta^1(n)} \otimes_{A \otimes^{\mathbf{L}} A}^{\mathbf{L}} (\mathbf{1} \otimes^{\mathbf{L}} \mathbf{1})$. \square

Corollary 6.9. *Let A be an algebra of bounded strict Tate-type and $e: A \rightarrow \mathbf{1}$ the corresponding augmentation. Let $A \rightarrow B \rightarrow \mathbf{1}$ be a factorization of e into a cofibration followed by a weak equivalence. Let B^\bullet be the cosimplicial algebra associated to $A \rightarrow B$. Then all algebras B^n are of bounded Tate-type. Moreover for any k the complexes $B^n(k)$ are uniformly in n cohomologically bounded from below.*

The same is true for the cobase change B'^\bullet of B^\bullet along $A \rightarrow B$.

Proof. By Corollary 6.7 and Lemma 6.8 the underlying homotopy type of B^1 is given as the total complex C of a bicomplex of the form

$$\mathbf{1} \leftarrow \overline{A} \leftarrow \overline{A}^{\otimes 2} \leftarrow \overline{A}^{\otimes 3} \leftarrow \cdots,$$

where \overline{A} sits in strictly negative Adams degree and is in each Adams degree cohomologically bounded from below. It follows that C has \mathbf{Z} in Adams degree 0 and is 0 in positive Adams degrees. Moreover the contribution to C in Adams degree $k < 0$ comes from at most $-k$ tensor factors of some $\overline{A}(l)$ with $l \geq k$. Using that $\mathrm{Tor}_i^{\mathbf{Z}}(M, N) = 0$ for $i > 1$, M, N abelian groups, it follows that C , and hence also B^1 , is in each Adams degree cohomologically bounded from below. This shows that B^1 is of Tate-type. By the Segal condition B^n is equivalent to $(B^1)^{\otimes n}$. Again it follows that B^n is contractible in positive Adams degrees, equivalent to \mathbf{Z} in Adams degree 0 and in each (negative) Adams degree cohomologically bounded from below. Hence B^n is of Tate-type. For $k > 0$ the contribution to $B^n(k)$ in $(B^1)^{\otimes n}$ only comes from at most $-k$ tensor factors of B^1 in Adams degrees l with $0 > l \geq k$, hence the $B^n(k)$ are uniformly in n cohomologically bounded from below.

The statement for the B'^\bullet follows from the fact that $B'^n \simeq B^{n+1}$. \square

Lemma 6.10. *Let M^\bullet be a cosimplicial object in $\mathrm{Cpx}(\mathbf{Ab})$ and suppose there is an $n_0 \in \mathbf{Z}$ such that for each $k \geq 0$ we have $H^n(M^k) = 0$ for $n < n_0$. Let $k_0 \geq 0$. Let C be the total complex corresponding to the double complex which is associated (via the unnormalized chain complex construction) to the truncated cosimplicial object $(M^k)_{k \leq k_0}$. Then the natural map $\mathrm{Tot}(M^\bullet) \rightarrow C$ induces isomorphisms $H^n(\mathrm{Tot}(M^\bullet)) \rightarrow H^n(C)$ for $n \leq k_0 + n_0$.*

Proof. By applying the good truncation functor there is a cosimplicial complex N^\bullet such that the complexes N^k are uniformly in k bounded from below (for the cohomological indexing) and a quasi isomorphism $M^\bullet \rightarrow N^\bullet$. Thus the map $\mathrm{Tot}(M^\bullet) \rightarrow \mathrm{Tot}(N^\bullet)$ is a quasi isomorphism. Consider the filtration on $\mathrm{Tot}(N^\bullet)$ induced by the subcomplexes coming from a cosimplicial degree bounded from below by a fixed degree. Then the associated spectral sequence strongly converges to the cohomology of $\mathrm{Tot}(N^\bullet)$ by the boundedness condition.

The strongly convergent spectral sequence for C coincides with this spectral sequence for fixed total degree $\leq k_0 + n_0$ by the assumption, thus the result. \square

We note that the notion of a (bounded) algebra of Tate type also makes sense in the category of Adams graded graded abelian groups. For example the cohomology of a (bounded) algebra of Tate type will be such a (bounded) algebra of Tate type.

Lemma 6.11. *a) Let A be a commutative algebra in Adams graded graded abelian groups which is of Tate type. Let M be an A -module which is Adams bounded from above (i.e. there is an N such that $M(k) = 0$ for $k > N$). Let $s \in \mathbf{N}_{>0}$. Suppose*

there is $B \in \mathbf{N}$ such that $A(k)^i = 0$ for $k \geq -s$ and $i < -B$ and suppose there is $B' \in \mathbf{Z}$ such that $M(k)^i = 0$ for $N \geq k \geq N - s$ and $i < B'$. Then there is an A -free resolution

$$\cdots \rightarrow P_i \rightarrow P_{i-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M$$

with the following properties:

- (1) if $i \geq 2s$ we have $P_i(N) = \cdots = P_i(N - s) = 0$,
- (2) for $2s > i \geq 0$ we have $P_i(k)^l = 0$ if $N \geq k \geq N - s$ and $l < B' - 2sB$,
- (3) for $2s > i \geq 0$ the generators of P_i which are in Adams degrees $N - s$ up to N lie all in degree $\geq B' - (2s - 1)B$.

b) Let the notation be as in a). Let M_2 be Adams bounded from above by N_2 and suppose there is $B'' \in \mathbf{Z}$ such that $M_2(k)^i = 0$ for $N \geq k \geq N - s$ and $i < B''$. Then $(M_2 \otimes_A P_i)(k)^l = 0$ if $N + N_2 \geq k \geq N + N_2 - s$ and $l < B' + B'' - (2s - 1)B$.

Proof. a) For an A -module M' we let $F(M')$ be the free A -module on the elements of A , i.e. for each $i, j \in \mathbf{Z}$ and $x \in M'(i)^j$ there is a copy $\Sigma^{i,j} A$ in $F(M')$. Inductively we construct P_i as follows: We set $P_0 := F(M)$. Note that the graded groups $P_0(N)^i, \dots, P_0(N - s)^i$ are zero for $i < B' - B$.

Let $K_0 = \ker(F(M) \rightarrow M)$. Then $K_0(N)$ is a graded abelian group consisting of free abelian groups (recall A is an Adams algebra and in particular $A(0)$ is the graded abelian group \mathbf{Z} sitting in degree 0). We let $Q = \bigoplus_{i \in \mathbf{Z}} \Sigma^{i,N} A \otimes K_0(N)^i$. There is a canonical map $Q \rightarrow K_0$ inducing an isomorphism in Adams degree N . Let K'_0 be the same module as K_0 except that the N -th Adams degree is set to 0. Let $Q' = F(K'_0)$. Then there is a canonical map $Q' \rightarrow K_0$. The induced map $Q \oplus Q' \rightarrow K_0$ is a surjection and we let $P = Q \oplus Q'$. Thus $P_1 \rightarrow P_0 \rightarrow M$ is exact.

We construct P_2 as $F(\ker(P_1 \rightarrow P_0))$. We construct P_3 as we constructed P_1 using the fact that $\ker(P_2 \rightarrow P_1)(N - 1)$ is a graded free abelian group. In this way we get a resolution $P_\bullet \rightarrow M$ such that the P_i are constructed in a 2-periodic way. The vanishing and boundedness results claimed are easy consequences of the construction of this resolution.

b) follows from item a)(3). \square

Lemma 6.12. *Let $A \rightarrow C$ be a map of bounded Tate algebras in $\mathbf{Cpx}(\mathbf{Ab})^{\mathbf{Z}}$ and let M^\bullet be a cosimplicial A -module. Let $L^\bullet := M^\bullet \otimes_A^{\mathbf{L}} C$. Suppose that there exists $N \in \mathbf{Z}$ such that $M^i(k) \simeq 0$ for all $i \geq 0$ and $k > N$ and that for any $k \leq N$ there exists $n_0 \in \mathbf{Z}$ such that $H^n(M^l(k)) = 0$ for all $n < n_0$ and all $l \geq 0$. Then L^\bullet satisfies the same boundedness conditions as M^\bullet (with possibly other constants) and the natural map $\mathrm{Tot}_A(M^\bullet) \otimes_A^{\mathbf{L}} C \rightarrow \mathrm{Tot}_C(L^\bullet)$ is an equivalence.*

Proof. We first note that it makes sense to talk about the total object of a truncated cosimplicial A -module, e.g. we can consider $\mathrm{Tot}(M^{\leq k})$. The same applies to L^\bullet . Clearly we have that the map

$$\mathrm{Tot}_A(M^{\leq k_0}) \otimes_A^{\mathbf{L}} C \rightarrow \mathrm{Tot}_C(L^{\leq k_0})$$

is an equivalence (because there are only finitely many contributions for each total degree). We let F denote the homotopy fiber of the map $\mathrm{Tot}_A(M^\bullet) \rightarrow \mathrm{Tot}_A(M^{\leq k_0})$. Fix $s \in \mathbf{N}_{>0}$. By assumption there exists $n_0 \in \mathbf{Z}$ such that $H^n(M^l(k)) = 0$ for $N \geq k \geq N - s$, $l \geq 0$ and $n < n_0$. By lemma (6.10) it follows that the map

$\mathrm{Tot}_A(M^\bullet) \rightarrow \mathrm{Tot}_A(M^{\leq k_0})$ induces an isomorphism in Adams degrees $N - s$ up to N and cohomological degree $\leq k_0 + n_0$. Thus in Adams degree $N - s$ up to N the cohomology of the fiber F vanishes in degrees $\leq k_0 + n_0$. Set $B'' := k_0 + n_0 + 1$.

We now invoke the strongly convergent Künneth spectral sequence for $F \otimes_A^{\mathbf{L}} C$:

$$E_{p,q,r}^2 = \mathrm{Tor}_p^{A_{**}}(F_{**}, C_{**})_{(q,r)} \implies (F \otimes_A^{\mathbf{L}} C)_{(p+q,r)}.$$

Here the $**$ -notation denotes homology. To compute the Tor-term we use the free resolution $P_\bullet \rightarrow C_{**}$ provided by lemma (6.11). It is then given by the homology of the complex $F_{**} \otimes_{A_{**}} P_\bullet$. The group $(F_{**} \otimes_{A_{**}} P_i)(k)^j$ contributes to

$$H^{j-i}((F \otimes_A^{\mathbf{L}} C)(k)).$$

We will be interested in these contributions only in Adams degrees from $N - s$ up to N . Let $B \in \mathbf{N}$ such that $H^i(A(k)) = 0$ for $0 \geq k \geq -s$ and $i < -B$ and let $B' \in \mathbf{Z}$ such that $H^i(C(k)) = 0$ for $0 \geq k \geq -s$ and $i < B'$. Then lemma (6.11)(b) tells us that $(F_{**} \otimes_{A_{**}} P_i)(k)^j = 0$, $N \geq k \geq N - s$, if either $i \geq 2s$ or $j < B' + B'' - (2s - 1)B$. These considerations give us that $H^i((F \otimes_A^{\mathbf{L}} C)(k)) = 0$ for $N \geq k \geq N - s$ and $i < B' + B'' - (2s - 1)B - 2s$.

Thus the map

$$\mathrm{Tot}_A(M^\bullet) \otimes_A^{\mathbf{L}} C \rightarrow \mathrm{Tot}_A(M^{\leq k_0}) \otimes_A^{\mathbf{L}} C$$

induces an isomorphism in Adams degree $N - s$ up to N in cohomological degree

$$(1) \quad < B' + k_0 + n_0 - (2s - 1)B - 2s.$$

For fixed s we can vary k_0 to increase this upper bound.

We now check the boundedness conditions for L^\bullet . Clearly it is also Adams bounded from above by N . The cohomological bounds for L^\bullet follow by an analogous spectral sequence argument as above.

We now consider the commutative square

$$\begin{array}{ccc} \mathrm{Tot}_A(M^\bullet) \otimes_A^{\mathbf{L}} C & \longrightarrow & \mathrm{Tot}_C(L^\bullet) \\ \downarrow & & \downarrow \\ \mathrm{Tot}_A(M^{\leq k_0}) \otimes_A^{\mathbf{L}} C & \longrightarrow & \mathrm{Tot}_C(L^{\leq k_0}). \end{array}$$

By what we have shown it follows that for fixed s all maps induce isomorphisms on cohomology in Adams degree $N - s$ up to N provided the cohomological degree is small. Increasing k_0 the explicit upper bound (1) shows that the upper horizontal map is a quasi isomorphism in these Adams degrees. But s was picked arbitrarily (the bounds n_0 , B and B' depend upon s), thus the result. \square

Lemma 6.13. *Let M^\bullet be a cosimplicial object in $\mathbf{Cpx}(\mathbf{Ab})$. Suppose there is an $n_0 \in \mathbf{Z}$ such that $H^n(M^i) = 0$ for every $n \leq n_0$ and $i \geq 0$. Then $\mathrm{Tot}(M^\bullet)$ is cohomologically bounded from below.*

Proof. Apply the good truncation to M^\bullet at the place n_0 : set $N^{i,n} = 0$ for $n < n_0$, $N^{i,n_0} = M^{i,n_0} / \ker(d)$ (d the differential $M^{i,n_0} \rightarrow M^{i,n_0+1}$) and $N^{i,n} = M^{i,n}$ for $n > n_0$. Then there is a quasi isomorphism $M^\bullet \rightarrow N^\bullet$. Clearly $\mathrm{Tot}(N^\bullet)$ is cohomologically bounded from below. But $\mathrm{Tot}(M^\bullet) \rightarrow \mathrm{Tot}(N^\bullet)$ is a quasi isomorphism. \square

In the following we suppose A is of bounded strict Tate type. We use the notations from the previous statements, i.e. we let $e: A \rightarrow \mathbf{1}$ be the corresponding augmentation and $A \rightarrow B \rightarrow \mathbf{1}$ a factorization of e into a cofibration followed by a weak equivalence.

We now introduce subcategories of $D(A)$ and $D(B^\bullet)$ which will restrict to an equivalence. Let $\mathcal{C} \subset D(A)_{\text{Aba}}$ be the full triangulated subcategory of modules which are cohomologically bounded from below in each Adams degree. Let $\mathcal{D} \subset D(B^\bullet)_{\text{cart}}$ be the full triangulated subcategory of modules M^\bullet , such that M^0 is Adams bounded from above and cohomologically bounded from below in each Adams degree.

Lemma 6.14. *Let $M^\bullet \in \mathcal{D}$. Then in each Adams degree M^n is uniformly in n cohomologically bounded from below.*

Proof. Since M^\bullet is cartesian we have $M^n \simeq M^0 \otimes^{\mathbf{L}} B^n$. By corollary (6.9) the property in question is fulfilled for B^\bullet . Suppose M is trivial in Adams degree $> k_0$. Fix $k \geq 0$. Then the contribution to Adams degree $k_0 - k$ in M^n is only from $M^0(k_0), \dots, M^0(k_0 - k)$ and $(B^n)(0), \dots, (B^n)(-k)$. The claim follows. \square

Lemma 6.15. *The canonical functor $D(A) \rightarrow D(B^\bullet)$ sends \mathcal{C} to \mathcal{D} .*

Proof. We have to check that the functor $e: D(A) \rightarrow D(\mathbf{1})$ given by push forward along the augmentation sends modules from \mathcal{C} to modules which are Adams bounded from above and in each Adams degree cohomologically bounded from below. Let $M \in \mathcal{C}$. We can write $e(M) = A \otimes_{A \otimes^{\mathbf{L}} A}^{\mathbf{L}} (M \otimes \mathbf{1})$. This in turn is the realization of a simplicial module $[n] \mapsto M \otimes^{\mathbf{L}} A^{\otimes^{\mathbf{L}} n} \otimes^{\mathbf{L}} \mathbf{1}$. As in lemma (6.8) this realization can be computed as the total complex C of a bicomplex

$$M \leftarrow M \otimes \overline{A} \leftarrow M \otimes \overline{A}^{\otimes 2} \leftarrow M \otimes \overline{A}^{\otimes 3} \leftarrow \dots$$

Suppose M is trivial in Adams degree $> k_0$. Fix $k \geq 0$. The module $M \otimes \overline{A}^{\otimes l}$ only contributes to Adams degree $k_0 - k$ in C if $l \leq k$ and then only the entries of \overline{A} in Adams degree $\geq -k$ and the $M(k_0), \dots, M(k_0 - k)$. This shows that C is cohomologically bounded from below in each Adams degree. \square

Lemma 6.16. *The canonical functor $D(B^\bullet) \rightarrow D(A)$ sends \mathcal{D} to \mathcal{C} .*

Proof. Let $M^\bullet \in \mathcal{D}$. Suppose M^0 is trivial in Adams degree $> k_0$. Then each M^n is trivial in Adams degree $> k_0$, since $M^n \simeq M^0 \otimes^{\mathbf{L}} B^n$ and B^n is trivial in positive Adams degree. Thus $\text{Tot}_A(M^\bullet)$ is trivial in Adams degree $> k_0$ which shows that the image of M^\bullet lies in $D(A)_{\text{Aba}}$.

By lemma (6.14) M^n is in each Adams degree cohomologically bounded from below, uniformly in n . Lemma (6.13) now shows that $\text{Tot}_A(M^\bullet)$ is in each Adams degree cohomologically bounded from below. \square

Lemma 6.17. *The subcategory \mathcal{C} satisfies the conditions of lemma (6.3).*

Proof. By lemma (6.1) property (3) of lemma (6.3) is fulfilled. Property (2) follows from lemmas (6.15) and (6.16). For property (1) we use lemma (6.12). The assumptions of lemma (6.12) are fulfilled by lemma (6.15) and lemma (6.14). This shows the claim. \square

Lemma 6.18. *The subcategory \mathcal{D} satisfies the conditions of lemma (6.4).*

Proof. Property (2) follows from lemmas (6.16) and (6.15). For property (1) we want to again apply lemma (6.12). The assumptions are fulfilled by lemma (6.14). This shows the claim. \square

Theorem 6.19. *Suppose A is of bounded Tate type. Then the adjunction*

$$\mathcal{D}(A) \rightleftarrows \mathcal{D}(B)_{\text{cart}}$$

restricts to an equivalence $\mathcal{C} \xrightarrow{\sim} \mathcal{D}$.

Proof. By lemmas (6.15) and (6.16) the adjunction restricts to an adjunction between \mathcal{C} and \mathcal{D} . By lemmas (6.17) and (6.18) we can apply lemmas (6.3) and (6.4) to conclude that the unit and counit of the induced adjunction are isomorphisms. This implies the claim. \square

We want to see that the equivalence between \mathcal{C} and \mathcal{D} further restricts to an equivalence between $\text{Perf}(A)$ and $\text{Perf}(B^\bullet)$. Define $\text{Perf}'(B^\bullet)$ to be the full triangulated subcategory of $\mathcal{D}(B^\bullet)_{\text{cart}}$ generated by the trivial one-dimensional representation B^\bullet . Then clearly $\text{Perf}'(B^\bullet) \subset \text{Perf}(B^\bullet)$, and the equivalence between \mathcal{C} and \mathcal{D} restricts to an equivalence $\text{Perf}(A) \xrightarrow{\sim} \text{Perf}'(B^\bullet)$. Note that in order to get this equivalence we would only have needed the first half of the above arguments about proving that the unit of the adjunction in question is an isomorphism.

The next result does not require B^\bullet to come from an algebra of bounded Tate type. Note that the categories $\text{Perf}(B^\bullet)$ and $\text{Perf}'(B^\bullet)$ make sense for arbitrary affine derived group schemes B^\bullet .

Proposition 6.20. *Suppose B^\bullet is an affine derived group scheme such that B^1 is an algebra of Tate type. Then we have $\text{Perf}'(B^\bullet) = \text{Perf}(B^\bullet)$.*

Proof. Let $M^\bullet \in \text{Perf}(B^\bullet)$, i.e. $M^\bullet \in \mathcal{D}(B^\bullet)_{\text{cart}}$ and M^0 is a perfect object in $\mathcal{D}(\mathbf{Ab})^{\mathbb{Z}}$. This means that M^0 only lives in finitely many Adams degrees and that the complex in each Adams degree is perfect. We can assume without loss of generality that M^\bullet is non-trivial. Let $\varphi: \mathcal{D}(\mathbf{1}) \rightleftarrows \mathcal{D}(B^\bullet)_{\text{cart}}: \psi$ be the adjunction where φ assigns to a module N the trivial B^\bullet -representation on N and ψ is the right adjoint of φ .

Let n_0 be the biggest Adams degree in which M^0 is non-trivial and m_0 the smallest such integers. We show that M^\bullet lies in the image of $\text{Perf}'(B^\bullet) \rightarrow \text{Perf}(B^\bullet)$ by induction on $n_0 - m_0$. Consider the counit $\varphi(\psi(M^\bullet)) \rightarrow M^\bullet$. We claim that the map $\varphi(\psi(M^\bullet))^0 \rightarrow M^0$ is an isomorphism in Adams degree n_0 .

Indeed, since B^1 is of Tate type, restricting the representation M^\bullet to the trivial group, i.e. pushing forward along $\mathcal{D}(B^\bullet)_{\text{cart}} \rightarrow \mathcal{D}(\mathbf{1}^\bullet)_{\text{cart}}$, does not change the Adams degree n_0 part of M^\bullet , and the counit for the adjunction $\mathcal{D}(\mathbf{1}) \rightleftarrows \mathcal{D}(\mathbf{1}^\bullet)_{\text{cart}}$ is an isomorphism.

We let $N(n_0) = \psi(M^\bullet)(n_0)$ and $N(n) = 0$ for $n \neq 0$. We let $f: N \rightarrow \psi(M^\bullet)$ be the canonical map in $\mathcal{D}(\mathbf{Ab})^{\mathbb{Z}}$. Then the composition

$$g: \varphi(N) \xrightarrow{\varphi(f)} \varphi(\psi(M^\bullet)) \rightarrow M^\bullet$$

is an isomorphism in Adams degree n_0 . By induction hypothesis (or in the case $n_0 = m_0$ since g is an isomorphism) the cofiber of g lies in the image of $\text{Perf}'(B^\bullet) \rightarrow \text{Perf}(B^\bullet)$, hence we are done. \square

Theorem 6.21. *Let A be of bounded Tate type. Then the canonical map $\text{Perf}(A) \rightarrow \text{Perf}(B^\bullet)$ is an equivalence.*

Proof. This follows from theorem (6.19) and proposition (6.20). \square

7. GENERAL COEFFICIENTS

In this section we want to generalize the previous results to general coefficients. For that let R be a commutative ring and let A be an algebra of bounded Tate type. We denote by A_R a (derived) base change of A to R , i.e. $A_R = A \otimes^{\mathbf{L}} R$. Let $A' \rightarrow A$ be the replacement which is of strict Tate type. Factor the augmentation $A' \rightarrow \mathbf{1}$ into a cofibration $A \rightarrow B$ followed by an equivalence $B \rightarrow \mathbf{1}$. Let $A \rightarrow B^\bullet$ be the coaugmented cosimplicial algebra associated to $A \rightarrow B$. Let $B_R^\bullet := B^\bullet \otimes^{\mathbf{L}} R$. Let

$$b_R: D(A) \rightleftarrows D(A_R): r_R$$

and

$$b'_R: D(B^\bullet) \rightleftarrows D(B_R^\bullet): r'_R$$

be the obvious adjunctions. Recall the subcategories $\mathbf{C} \subset D(A)$ and $\mathbf{D} \subset D(B^\bullet)_{\text{cart}}$ defined in the previous section. We let $\mathbf{C}_R \subset D(A_R)$ be the full subcategory of objects X such that $r_R(X) \in \mathbf{C}$ and $\mathbf{D}_R \subset D(B_R^\bullet)_{\text{cart}}$ the full subcategory of objects X such that $r'_R(X) \in \mathbf{D}$.

Theorem 7.1. *Suppose A is of bounded Tate type. The adjunction $D(A_R) \rightleftarrows D(B_R^\bullet)$ restricts to an equivalence $\mathbf{C}_R \rightarrow \mathbf{D}_R$.*

Proof. The diagrams

$$\begin{array}{ccc} D(A_R) & \longrightarrow & D(B_R^\bullet) \\ \downarrow r_R & & \downarrow r'_R \\ D(A) & \longrightarrow & D(B^\bullet) \end{array}$$

and

$$\begin{array}{ccc} D(A_R) & \longleftarrow & D(B_R^\bullet) \\ \downarrow r_R & & \downarrow r'_R \\ D(A) & \longleftarrow & D(B^\bullet) \end{array}$$

are 2-commutative. Moreover the functors r_R and r'_R detect isomorphisms. The claim follows then from theorem (6.19). \square

Let $\text{Perf}(A_R) \subset D(A_R)$ be the full subcategory of perfect A_R -modules and let $\text{Perf}(B_R^\bullet) \subset D(B_R^\bullet)_{\text{cart}}$ be the full subcategory of B_R^\bullet -modules M^\bullet such that each M^n is a perfect B_R^n -module, or equivalently such that M^0 is a perfect $B^0 \simeq R$ -module.

Note that $\text{Perf}(A_R) \subset \mathbf{C}_R$ and $\text{Perf}(B_R^\bullet) \subset \mathbf{D}_R$ (for the first inclusion we use that $\text{Tor}_i^{\mathbf{Z}}(M, N) = 0$ for $i \geq 2$, M and N abelian groups).

Theorem 7.2. *Suppose A is of bounded Tate type. Then the natural functor $\text{Perf}(A_R) \rightarrow \text{Perf}(B_R^\bullet)$ is an equivalence.*

Proof. By theorem (7.1) we are left to prove that the functor in question is essentially surjective. This works exactly as in the proof of proposition (6.20). \square

We now turn to the case where A is given over a commutative base ring R . So suppose A is a commutative \mathbb{S} -algebra in $\text{Cpx}(R)^{\mathbb{Z}}$. The definition of Tate type, strict Tate Type and bounded Tate type works as in the case over the integers.

We suppose from now on that A is of bounded Tate type. Without loss of generality we can assume that A is also of strict Tate type.

Factor the canonical augmentation $A \rightarrow \mathbf{1}_R$ into a cofibration $A \rightarrow B$ followed by a weak equivalence $B \rightarrow \mathbf{1}_R$. Let $A \rightarrow B^\bullet$ be the coaugmented cosimplicial algebra associated to $A \rightarrow B$. As in the case over the integers we can define the full subcategories $\mathcal{C} \subset \mathcal{D}(A)$ and $\mathcal{D} \subset \mathcal{D}(B^\bullet)_{\text{cart}}$.

Theorem 7.3. *Suppose A is of bounded Tate type and that R has finite homological dimension. Then the adjunction*

$$\mathcal{D}(A) \rightleftarrows \mathcal{D}(B)_{\text{cart}}$$

restricts to an equivalence $\mathcal{C} \xrightarrow{\sim} \mathcal{D}$.

Proof. The steps are as for theorem (6.19). Our assumption on R is needed in the analogues of corollary (6.9), lemma (6.11), lemma (6.12), lemma (6.14), lemma (6.15) and lemma (6.16).

All other steps are exactly as for theorem (6.19). \square

As in the case over the integers we define subcategories $\text{Perf}(A) \subset \mathcal{D}(A)$ and $\text{Perf}(B^\bullet) \subset \mathcal{D}(B^\bullet)_{\text{cart}}$.

Theorem 7.4. *Suppose A is of bounded Tate type and that R has finite homological dimension. Then the natural functor $\text{Perf}(A) \rightarrow \text{Perf}(B^\bullet)$ is an equivalence.*

Proof. By theorem 7.3 we are left to prove that the functor in question is essentially surjective. This works exactly as in the proof of proposition 6.20. \square

Remark 7.5. *It is possible to generalize theorems (7.1) and (7.2) to the relative case where we are given a map of commutative algebras $R \rightarrow R'$ and R satisfies the assumption of theorem (7.4). We start with an algebra A of bounded Tate type over R as above and build $A_{R'} = A \otimes_R^{\mathbf{L}} R'$ and $B_{R'}^\bullet = B^\bullet \otimes_R^{\mathbf{L}} R'$. Then the corresponding versions of theorems (7.1) and (7.2) are still valid.*

8. REPRESENTATIONS OF \mathbb{G}_m

In the whole section we work with strictly commutative algebras and the usual notion of module (in contrast to the possibility that we view a strictly commutative algebra as an E_∞ -algebra and work with the corresponding notion of module).

We denote by G the group scheme \mathbb{G}_m over \mathbf{Z} . Let $A = \mathbf{Z}[z^{\pm 1}]$ be the corresponding Hopf algebra and A^n its n -th tensor power as an algebra. The A^n assemble to the cosimplicial algebra A^\bullet in $\text{Cpx}(\mathbf{Ab})$ modelling G .

We denote by $\text{Rep}^{\text{str}}(G)$ the category of strict representations of G with values in $\text{Cpx}(\mathbf{Ab})$, i.e. the category of complexes of A -comodules. Also we denote by $\text{Mod}(A^\bullet)$ the category of complexes of A^\bullet -modules.

There is an embedding $\iota^{\text{str}} : \text{Rep}^{\text{str}}(G) \rightarrow \text{Mod}(A^\bullet)$ whose image is the subcategory of strictly cartesian A^\bullet -modules.

Let $j_1 : \text{Cpx}(\mathbf{Ab})^{\mathbf{Z}} \rightarrow \text{Rep}^{\text{str}}(G)$ be the functor which assigns to a complex M sitting totally in Adams degree r the unique representation of G on M with weight r and which commutes with sums.

So the value of j_1 on $(M(r))_{r \in \mathbf{Z}}$ is $\bigoplus_{r \in \mathbf{Z}} M(r) \otimes \mathbf{Z}(r)$, where we denote $\mathbf{Z}(r)$ the weight r representation of G on \mathbf{Z} .

Denote by j the composition $\iota^{\text{str}} \circ j_1$. Note this is a tensor functor.

Clearly we have a derived functor

$$\mathbf{L}j : \text{D}(\mathbf{Ab})^{\mathbf{Z}} \rightarrow \text{D}(A^\bullet)$$

which factors as

$$\mathbf{L}j : \text{D}(\mathbf{Ab})^{\mathbf{Z}} \rightarrow \text{D}(A^\bullet)_{\text{cart}}.$$

The functor j has a right adjoint u given by

$$u(F^\bullet)(r)_m = \text{Hom}_{\text{Mod}(A^\bullet)}(j(D^{m,r}(\mathbf{Z})), F^\bullet).$$

To show that $\mathbf{L}j$ has a right adjoint it is easiest to show that j is a left Quillen functor for adequate model structures. On $\text{Cpx}(\mathbf{Ab})^{\mathbf{Z}}$ we take the usual projective model structure. The model structure on $\text{Mod}(A^\bullet)$ should fulfill the property that levelwise (for the cosimplicial direction) projective cofibrations are cofibrations, which for example is fulfilled by the injective model structure (i.e. a map is a cofibration if and only if it is a monomorphism). Hence it follows that u has a right derived functor $\mathbf{R}u$ which is right adjoint to $\mathbf{L}j$. Note that for this argument we do not need to use a symmetric monoidal model structure on $\text{Mod}(A^\bullet)$.

We first give a proof of the following fact.

Lemma 8.1. *The restriction of $\mathbf{L}j$ to perfect objects gives a fully faithful embedding of tensor triangulated categories*

$$(\text{D}(\mathbf{Ab})^{\mathbf{Z}})^{\text{perf}} \rightarrow \text{Perf}(A^\bullet).$$

Before starting the proof we give an explicit description of the cosimplicial algebra A^\bullet and of the cosimplicial A^\bullet -module M^\bullet corresponding to a representation of pure weight r on a complex $M \in \text{Cpx}(\mathbf{Ab})$.

First we fix our conventions for simplicial and cosimplicial objects.

For $n > 0$ and $0 \leq i \leq n$ we denote by d_i the unique strictly monotone map $[n-1] \rightarrow [n]$ which omits i in the target.

Now let A be any commutative Hopf algebra (over some ground ring, which we omit in the notation) with comultiplication $\nabla : A \rightarrow A \otimes A$ and associated cosimplicial algebra A^\bullet .

The cosimplicial maps $d_i^A : A^{\otimes(n-1)} \rightarrow A^{\otimes n}$ are given as

$$(2) \quad d_i^A(a_1 \otimes \cdots \otimes a_{n-1}) = \begin{cases} 1 \otimes a_1 \otimes \cdots \otimes a_{n-1} & i = 0 \\ a_1 \otimes \cdots \otimes a_{i-1} \otimes \nabla(a_i) \otimes \cdots \otimes a_{n-1} & 0 < i < n \\ a_1 \otimes \cdots \otimes a_{n-1} \otimes 1 & i = n \end{cases}$$

Now let $c : M \rightarrow M \otimes A$ be a right A -comodule structure on an object $M \in \mathbf{Cpx}(\mathbf{Ab})$. Then we get an associated A^\bullet -module M^\bullet as follows: We have $M^n = M \otimes A^{\otimes n}$ and the cosimplicial map $d_i^M : M^{n-1} \rightarrow M^n$ lying over d_i^A is defined as (3)

$$d_i^M(m \otimes a_1 \otimes \cdots \otimes a_{n-1}) = \begin{cases} c(m) \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes m & i = 0 \\ m \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes \nabla(a_i) \otimes \cdots \otimes a_{n-1} & 0 < i < n \\ m \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes 1 & i = n \end{cases}$$

Specializing to $A = \mathbf{Z}[z^{\pm 1}]$ we get the following description of A^\bullet :

- $A^n = \mathbf{Z}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$,
- $d_0^A(z_l) = z_{l+1}, l = 1, \dots, n-1$,
- for $0 < i < n$:
 - $d_i^A(z_l) = z_l, l \leq i-1$,
 - $d_i^A(z_i) = z_i z_{i+1}$,
 - $d_i^A(z_l) = z_{l+1}, i+1 \leq l \leq n-1$,
- $d_n^A(z_l) = z_l, l = 1, \dots, n-1$.

The cosimplicial module

$$(4) \quad \mathbf{Z}(r)^\bullet$$

corresponding to the weight r representation $\mathbf{Z}(r)$ has $M^n = A^n$, $d_0^{\mathbf{Z}(r)}(b) = d_0^A(b) \cdot z_0^r$, all other coface maps are the same as for A^\bullet .

Proof of lemma 8.1. The full subcategory of perfect objects of $\mathbf{D}(\mathbf{Ab})^{\mathbf{Z}}$ is generated by objects of the form $\mathbf{Z}(r)$, $r \in \mathbf{Z}$, hence to prove fully faithfulness of $\mathbf{L}j$ on perfect objects it is sufficient to see that the map

$$(5) \quad \mathrm{Hom}_{\mathbf{D}(\mathbf{Ab})^{\mathbf{Z}}}(\mathbf{Z}, \mathbf{Z}(r)[k]) \rightarrow \mathrm{Hom}_{\mathbf{D}(A^\bullet)}(\mathbf{Z}, \mathbf{Z}(r)[k])$$

is an isomorphism for $r, k \in \mathbf{Z}$. The left hand side of (5) is \mathbf{Z} if $r = k = 0$, otherwise it is 0. For fixed r the right hand side is computed by the complex associated to the cosimplicial abelian group $\mathbf{Z}(r)^\bullet$, see (4). We describe the corresponding normalized chain complex obtained by dividing out the images of the $d_i^{\mathbf{Z}(r)}$, $0 \leq i \leq n-1$, in $\mathbf{Z}(r)^n = A^n$ and using $\partial_n := (-1)^n d_n^{\mathbf{Z}(r)} : A^{n-1}/\{\text{images}\} \rightarrow A^n/\{\text{images}\}$ as differential. The differentials d_i respect the direct sum decompositions of the A^n by the monomials in the z_i , $i = 1, \dots, n$. Let $n > 0$. A monomial $z_1^{e_1} \cdots z_n^{e_n} \in A^n$ lies in the image of one of the d_i , $i = 0, \dots, n-1$, if and only if $e_1 = r$ or $e_j = e_{j+1}$ for some $1 \leq j \leq n-1$. Suppose a non-zero monomial $z_1^{e_1} \cdots z_n^{e_n} \in A^n/\{\text{images}\}$ is mapped to 0 by the differential ∂_{n+1} . This is the case if and only if $e_n = 0$, i.e. if it is in the image of the differential ∂_n . So the cohomology of the normalized chain complex of $\mathbf{Z}(r)^\bullet$ is 0 in positive degrees. The differential $\partial_1 : A^0 \rightarrow A^1/\mathrm{im}(d_0)$ is 0 if $r = 0$ and injective otherwise, hence we see that the map (5) is indeed an isomorphism. \square

Proposition 8.2. *The functor $\mathbf{L}j$ is a full embedding.*

Proof. By lemma 8.1 we know that $\mathbf{L}j$ is an embedding on perfect objects. Since $\mathbf{D}(\mathbf{Ab})^{\mathbf{Z}}$ is generated as triangulated category by objects $M = \bigoplus_{i \in I} \mathbf{Z}(k_i)[l_i]$ we are finished if

$$(6) \quad \bigoplus_{i \in I} \mathrm{Hom}_{\mathbf{D}(A^\bullet)}(\mathbf{Z}, \mathbf{Z}(k_i)[l_i]) \rightarrow \mathrm{Hom}_{\mathbf{D}(A^\bullet)}(\mathbf{Z}, \bigoplus_{i \in I} \mathbf{Z}(k_i)[l_i])$$

is an isomorphism. We denote the A^\bullet -module corresponding to M by M^\bullet . By theorem 8.1 the left hand side of 6 is

$$(7) \quad \bigoplus_{i \in I, k_i = l_i = 0} \mathbf{Z}.$$

The right hand side of (6) is given as H_0 of the (derived) total object $\text{Tot}(M^\bullet)$ of the cosimplicial complex M^\bullet . It is known that the total object is given as the total complex of the double complex associated to M^\bullet , where for the value of the total complex in cohomological degree n one has to use the *product* $\prod_{p+q=n} M^{p,q}$. (One can also use the normalized associated complex.) We denote this total complex by $\text{Tot}(M^\bullet)$. The only differential in this total complex comes from the cosimplicial structure. Hence $\text{Tot}(M^\bullet)$ splits as a product of complexes $\prod_{n \in \mathbf{Z}} \text{Tot}(M_n^\bullet)$, where M_n^\bullet is the A^\bullet -module corresponding to $\bigoplus_{i \in I, l_i = n} \mathbf{Z}(k_i)[l_i]$. By exactness of products in **Ab** we get that the right hand side of (6) is

$$(8) \quad \prod_{n \in \mathbf{Z}} H_0(\text{Tot}(M_n^\bullet)) = H_0(\text{Tot}(M_0^\bullet)) = \bigoplus_{i \in I, k_i = l_i = 0} \mathbf{Z},$$

and it is easy to see that the map (6) is the identity on the identifications (7) and (8). \square

Lemma 8.3. *Every homotopy cartesian A^\bullet -module M^\bullet such that the cohomology of M^0 is bounded lies in the essential image of $\mathbf{L}j$.*

Proof. Without loss of generality we can assume that $H^i(M^0) = 0$ for $i < 0$ and if M^0 is non-contractible that $H^0(M^0) \neq 0$. Let l be the largest integer such that $H^l(M^0) \neq 0$ if M^0 is non-contractible, otherwise we set $l = -1$. We prove the statement by induction on l .

The start of the induction is $l = -1$ where we have nothing to prove. So $l \geq 0$. By replacing $M^{n,0}$ by $M^{n,0}/\text{image}(M^{n,-1})$ and setting $M^{n,i} = 0$ for $i < 0$ we can assume that the complexes M^n sit purely in cohomologically non-negative degrees. The cycles $Z^n = \ker(M^{n,0} \rightarrow M^{n,1})$ equal the cohomology $H^0(M^n)$ and form a strictly cartesian A^\bullet -module. They thus form a direct sum of representations of \mathbb{G}_m of the form $M_r(r)$ for abelian groups M_r . Hence Z^\bullet lies in the image of $\mathbf{L}j$. Applying the induction hypothesis to the (shifted) cofiber of $Z^\bullet \rightarrow M^\bullet$ we see that M^\bullet itself is a cofiber of objects from the image of $\mathbf{L}j$. Using fully faithfulness (proposition (8.2)) yields the induction step. \square

Proposition 8.4. *The functor $\mathbf{L}j: \mathbf{D}(\mathbf{Ab})^{\mathbf{Z}} \rightarrow \mathbf{D}(A^\bullet)_{\text{cart}}$ is essentially surjective.*

Proof. Let M^\bullet be a homotopy cartesian A^\bullet -module. We will apply levelwise good truncation functors to M^\bullet . So for $S \in \{\leq a, \geq a, [a, b]\}$ denote by $\tau^S(M^\bullet)$ the good truncation such that the cohomology is preserved in the indicated region and is 0 otherwise. We have an exact triangle

$$\tau^{\leq 0}(M^\bullet) \rightarrow M^\bullet \rightarrow \tau^{\geq 1}(M^\bullet) \rightarrow \tau^{\leq 0}(M^\bullet)[1]$$

in $\mathbf{D}(A^\bullet)_{\text{cart}}$. To prove that M^\bullet is in the essential image of $\mathbf{L}j$ it is thus sufficient by proposition (8.2) that $\tau^{\leq 0}(M^\bullet)$ and $\tau^{\geq 1}(M^\bullet)$ are in the essential image of $\mathbf{L}j$.

Let us do first the case of $\tau^{\leq 0}(M^\bullet)$: We can write

$$\tau^{\leq 0}(M^\bullet) \simeq \text{holim}_{a \rightarrow -\infty} \tau^{[a, 0]}(M^\bullet).$$

Now the functor $\mathbf{R}u: \mathbf{D}(A^\bullet) \rightarrow \mathbf{D}(\mathbf{Ab})^{\mathbb{Z}}$ preserves homotopy limits, so we have

$$\mathbf{R}u(\tau^{\leq 0}(M^\bullet)) \simeq \operatorname{holim}_{a \rightarrow -\infty} \mathbf{R}u(\tau^{[a,0]}(M^\bullet)).$$

By lemma (8.3) the i -th cohomology of $\mathbf{R}u(\tau^{\leq 0}(M^\bullet))$ is thus given by $H^i(M^\bullet)$ for $i \leq 0$ (viewing the \mathbb{G}_m -representation $H^i(M^\bullet)$ as a graded object). The commutativity of the diagram

$$\begin{array}{ccc} \mathbf{L}j\mathbf{R}u(\tau^{\leq 0}(M^\bullet)) & \longrightarrow & \tau^{\leq 0}(M^\bullet) \\ \downarrow & & \downarrow \\ \mathbf{L}j\mathbf{R}u(\tau^{[a,0]}(M^\bullet)) & \longrightarrow & \tau^{[a,0]}(M^\bullet) \end{array}$$

now shows that the upper horizontal map is an equivalence.

We turn to showing that $\tau^{\geq 1}(M^\bullet)$ lies in the essential image of $\mathbf{L}j$. We can write

$$\tau^{\geq 1}(M^\bullet) \simeq \operatorname{hocolim}_{a \rightarrow \infty} \tau^{[1,a]}(M^\bullet).$$

Let $X_a := \mathbf{R}u(\tau^{[1,a]}(M^\bullet))$ and set $X := \operatorname{hocolim}_{a \rightarrow \infty} X_a$. Then

$$\mathbf{L}j(X) = \operatorname{hocolim}_{a \rightarrow \infty} \mathbf{L}j(X_a)$$

and we have a canonical map $\mathbf{L}j(X) \rightarrow \tau^{\geq 1}(M^\bullet)$. By lemma (8.3) this is an isomorphism on cohomology hence an equivalence. This finishes the proof of the essential surjectivity of $\mathbf{L}j$. \square

Theorem 8.5. *The functor $\mathbf{L}j: \mathbf{D}(\mathbf{Ab})^{\mathbb{Z}} \rightarrow \mathbf{D}(A^\bullet)_{\text{cart}}$ is an equivalence of tensor triangulated categories.*

Proof. This combines propositions (8.2) and (8.4). \square

Corollary 8.6. *Let R be a commutative \mathbb{S} -algebra in $\mathbf{Cpx}(\mathbf{Ab})^{\mathbb{Z}}$. Let R' be its image in $\mathbf{Mod}(A^\bullet)$ under j . Then the induced functor $\mathbf{D}(R) \rightarrow \mathbf{D}(R')_{\text{cart}}$ is an equivalence of tensor triangulated categories.*

Proof. This follows from theorem (8.5) and the fact that the adjoint functors $\mathbf{D}(R) \rightleftarrows \mathbf{D}(R')_{\text{cart}}$ commute with forgetting the R - resp. R' -module structure. \square

9. TRANSFER ARGUMENT

We keep the notation from section (8), so A^\bullet denotes the cosimplicial algebra corresponding to \mathbb{G}_m . We consider a cosimplicial commutative \mathbb{S} -algebra B^\bullet in $\mathbf{Cpx}(\mathbf{Ab})^{\mathbb{Z}}$ and denote $C^\bullet = j(B^\bullet)$, i.e. C^\bullet is a cosimplicial algebra in $\mathbf{Mod}(A^\bullet)$. In particular C^\bullet is a bicosimplicial commutative \mathbb{S} -algebra in $\mathbf{Cpx}(\mathbf{Ab})$.

There is a functor $j_B: \mathbf{Mod}(B^\bullet) \rightarrow \mathbf{Mod}(C^\bullet)$ which has a left derived functor $\mathbf{L}j_B$. Both j_B and $\mathbf{L}j_B$ are tensor functors.

As j the functor j_B has a right adjoint, which we denote by u_B . Using model structure one sees again that u_B has a right derived functor $\mathbf{R}u_B$.

We denote by $\mathbf{D}(C^\bullet)_{\text{cart}}$ the homotopy category of C^\bullet -modules which are homotopy cartesian as bicosimplicial modules.

Corollary 9.1. *The natural functor $\mathbf{L}j_B: \mathbf{D}(B^\bullet)_{\text{cart}} \rightarrow \mathbf{D}(C^\bullet)_{\text{cart}}$ is an equivalence of tensor triangulated categories.*

Proof. This follows from corollary (8.6) and the fact that the adjoint functors in question commute with the functors restricting a cosimplicial module to one level. \square

Recall the maps $\alpha_i: [1] \rightarrow [n]$ defined before definition (5.1).

Lemma 9.2. *Let $n > 0$. Let R^\bullet be an affine derived group scheme in $\mathbf{Cpx}(\mathbf{Ab})$ over \mathbf{Z} and let S^\bullet be a homotopy cartesian commutative R^\bullet - \mathbb{S} -algebra. Then the map $(S^1)^{\otimes_{R^\bullet} n} \rightarrow (S^n)^{\otimes_{R^\bullet} n}$ induced by the maps $\alpha_0, \dots, \alpha_{n-1}$ is an equivalence.*

Proof. We let $R' = (R^1)^{\otimes_{R^\bullet} n}$ and $\beta_i: R^1 \rightarrow R'$ the i -th inclusion, $1 \leq i \leq n$. Let S_i be the push forward of S^1 with respect to β_i . Then the map

$$(S^1)^{\otimes_{R^\bullet} n} \rightarrow S_1 \otimes_{R'}^{\mathbf{L}} \dots \otimes_{R'}^{\mathbf{L}} S_n$$

induced by the natural maps $S^1 \rightarrow S_i$ is an equivalence.

There are maps $S_i \rightarrow S^n$ induced by the maps $(\alpha_{i-1})_*: S^1 \rightarrow S^n$ lying over the map $(R^1)^{\otimes_{R^\bullet} n} \rightarrow R^n$ induced by the $\alpha_0, \dots, \alpha_{n-1}$.

The latter map is an equivalence since R^\bullet is an affine derived group scheme. Thus the induced map

$$S_1 \otimes_{R'}^{\mathbf{L}} \dots \otimes_{R'}^{\mathbf{L}} S_n \rightarrow (S^n)^{\otimes_{R^\bullet} n}$$

is also an equivalence. This shows the claim. \square

We keep the notations from this paragraph. We assume now that B^\bullet represents an affine derived group scheme over $\mathbf{1}$, i.e. fulfills the conditions of definition 5.1. Then C^\bullet also represents an affine derived group scheme in $\mathbf{Mod}(A^\bullet)$.

We denote by D^\bullet the diagonal of the bicosimplicial commutative \mathbb{S} -algebra underlying C^\bullet .

Lemma 9.3. *The cosimplicial commutative \mathbb{S} -algebra D^\bullet in $\mathbf{Cpx}(\mathbf{Ab})$ is an affine derived group scheme over \mathbf{Z} .*

Proof. We denote by $C^{n,k}$ the entry in cosimplicial degree k of the A^\bullet -algebra C^n . Since B^\bullet is an affine derived group scheme we know that the map

$$C^{1,k} \otimes_{A^k}^{\mathbf{L}} \dots \otimes_{A^k}^{\mathbf{L}} C^{1,k} \rightarrow C^{k,k}$$

induced by the maps $\alpha_0, \dots, \alpha_{k-1}$ in the first cosimplicial direction is an equivalence.

Furthermore by lemma (9.2) the map

$$(C^{1,1})^{\otimes_{A^k} \mathbf{L} k} \rightarrow (C^{1,k})^{\otimes_{A^k} \mathbf{L} k}$$

induced by the α_i in the second cosimplicial direction is an equivalence. This establishes that also the composite map

$$(C^{1,1})^{\otimes_{A^k} \mathbf{L} k} \rightarrow C^{k,k}$$

induced by the α_i in the diagonal cosimplicial direction is an equivalence. This shows the first property necessary for being an affine derived group scheme.

The proof for the second condition for being an affine derived group scheme is similar:

As in lemma (9.2) it follows that the map

$$C^{1,1} \otimes^{\mathbf{L}} C^{1,1} \rightarrow C^{1,2} \otimes_{A_2}^{\mathbf{L}} C^{1,2}$$

induced by the maps c, α_0 in the second cosimplicial direction is an equivalence since A^\bullet satisfies the second condition of being an affine derived group scheme and since $C^{1,\bullet}$ is homotopy cartesian over A^\bullet .

Furthermore the map

$$C^{1,2} \otimes_{A_2}^{\mathbf{L}} C^{1,2} \rightarrow C^{2,2}$$

induced by c, α_0 in the first cosimplicial direction is an equivalence since B^\bullet satisfies the second condition of being an affine derived group scheme.

Again the composite of these two maps yields the map which the second condition of being an affine derived group scheme requests to be an equivalence.

The third condition that the map $\mathbf{Z} \rightarrow D^0$ is an equivalence is satisfied since $A^0 = \mathbf{Z}$ and the map $\mathbf{1} \rightarrow C^0$ is an equivalence.

This finishes the proof of the claim. \square

Morally we can think of D^\bullet as the derived affine group scheme

$$(9) \quad B^\bullet \rtimes \mathbb{G}_m.$$

We will use this notation in the section on examples.

Proposition 9.4. *The restriction functor $D(C^\bullet)_{\text{cart}} \rightarrow D(D^\bullet)_{\text{cart}}$ is an equivalence of categories.*

Proof. This follows from corollary (4.11). \square

Theorem 9.5. *There is a natural equivalence of tensor triangulated categories $D(B^\bullet)_{\text{cart}} \rightarrow D(D^\bullet)_{\text{cart}}$. It restricts to an equivalence $\text{Perf}(B^\bullet) \rightarrow \text{Perf}(D^\bullet)$.*

Proof. This combines corollary (9.1) and proposition (9.4). \square

We are going to describe the above result in the case of algebras over a given commutative ring R .

Set $A_R^\bullet = A^\bullet \otimes R$. Let B^\bullet be an affine derived group scheme in $\text{Cpx}(R)^{\mathbf{Z}}$. Let $C^\bullet = j(B^\bullet)$. So C^\bullet is a derived affine group scheme in $\text{Mod}(A_R^\bullet)$. In particular it is a bicosimplicial commutative \mathbb{S} -algebra in $\text{Cpx}(R)$. Let D^\bullet be its diagonal.

Theorem 9.6. *There is a natural equivalence of tensor triangulated categories $D(B^\bullet)_{\text{cart}} \rightarrow D(D^\bullet)_{\text{cart}}$. It restricts to an equivalence $\text{Perf}(B^\bullet) \rightarrow \text{Perf}(D^\bullet)$.*

Proof. This combines the R -analogues of corollary (9.1) and proposition (9.4). \square

Proof of theorem (2.2). If the characteristic of k is 0 then by [17, Corollary 6.9] there exists a commutative \mathbb{S} -algebra A in $\text{Cpx}(\mathbf{Ab})^{\mathbf{Z}}$ such that $D(A)$ is naturally equivalent to $\text{DMT}(X)$ as tensor triangulated category. In general this follows from the fact that over any scheme which lives over a field the Eilenberg MacLane spectrum \mathbf{MZ} is strongly periodizable in the language of [17]. This follows from [17, Corollary 6.3] for perfect fields, in general one uses the fact that there is a map of E_∞ -ring spectra from the pullback of the Eilenberg MacLane spectrum over the prime field to the Eilenberg MacLane spectrum over the given base.

Now \mathbf{MZ} receives a canonical map from the push forward of the topological Eilenberg MacLane spectrum. This implies that there is an \mathbb{S} -algebra A in $\mathbf{Cpx}(\mathbf{Ab})^{\mathbb{Z}}$ such that $D(A)$ is naturally equivalent to $\mathbf{DMT}(X)$ as tensor triangulated category by a representation theorem similar to [17, Corollary 6.9].

The equivalence $\mathbf{DMT}(X) \simeq D(A)$ restricts to an equivalence $\mathbf{DMT}_{\text{gm}}(X) \simeq \text{Perf}(A)$. Note that A is of bounded Tate type by our assumptions on $\mathbf{DMT}(X)$. We replace A by the canonical algebra of strict Tate type and denote it again by A . Factor the canonical augmentation $A \rightarrow \mathbf{1}$ into a cofibration $A \rightarrow B$ followed by a weak equivalence $B \rightarrow \mathbf{1}$. Let $A \rightarrow B^\bullet$ the coaugmented cosimplicial algebra associated to $A \rightarrow B$. Then by proposition (5.3) B^\bullet is an affine derived group scheme over $\mathbf{1}$. Theorem (6.21) gives us an equivalence of tensor triangulated categories $\text{Perf}(A) \simeq \text{Perf}(B^\bullet)$.

We next apply the procedure of this paragraph. So let $C^\bullet = j(B^\bullet)$ and let D^\bullet be the diagonal of C^\bullet . Then theorem (9.5) gives an equivalence of tensor triangulated categories $\text{Perf}(B^\bullet) \simeq \text{Perf}(D^\bullet)$. So D^\bullet is our looked for affine derived group scheme with the property $\mathbf{DMT}_{\text{gm}}(X) \simeq \text{Perf}(D^\bullet)$.

For the case with R -coefficients observe first that there is an equivalence

$$\mathbf{DMT}_{\text{gm}}(X)_R \simeq \text{Perf}(A_R).$$

Next theorem (7.2) gives the equivalence $\text{Perf}(A_R) \simeq \text{Perf}(B_R^\bullet)$. Theorem (9.6) yields $\text{Perf}(B_R^\bullet) \simeq \text{Perf}(D_R^\bullet)$ which gives the conclusion. \square

Proof of theorem (2.3). The proof goes along the same lines as the proof of theorem (2.2) using remark (7.5). \square

Proof of theorem (2.1). As in the proof of theorem (2.2) we can find a Tate-algebra A in $\mathbf{Cpx}(\mathbb{Q})^{\mathbb{Z}}$ such that $\mathbf{DMT}(X)_{\mathbb{Q}} \simeq D(A)$ and $\mathbf{DMT}_{\text{gm}}(X)_{\mathbb{Q}} \simeq \text{Perf}(A)$. Now we claim that we can write A as a filtered (homotopy) colimit of *bounded* Tate-algebras, $A \simeq \text{colim}_i A_i$. That granted we apply the procedure used in the proof of theorem (2.2) to get affine derived group schemes D_i^\bullet such that $\text{Perf}(A_i) \simeq \text{Perf}(D_i^\bullet)$. The D_i^\bullet can be defined in such a way that they depend functorially on i . In this way we get a pro affine derived group scheme “ \lim_i ” D_i^\bullet . Noting that $\text{Perf}(A) \simeq 2\text{-colim}_i \text{Perf}(A_i)$ we get equivalences

$$\begin{aligned} \text{Perf}(\text{“}\lim_i\text{”} D_i^\bullet) &= 2\text{-colim}_i \text{Perf}(D_i^\bullet) \\ &\simeq 2\text{-colim}_i \text{Perf}(A_i) \simeq \text{Perf}(A) \simeq \mathbf{DMT}_{\text{gm}}(X)_{\mathbb{Q}} \end{aligned}$$

which was to be shown.

It remains to prove that we can write A as a filtered (homotopy) colimit of bounded Tate algebras. We give two procedures. The first one (mentioned in the introduction) writes A as a filtered (homotopy) colimit of finite type cell Tate-algebras. So we have to prove that such a a finitely generated Tate-algebra is bounded. Here we use that we work over \mathbb{Q} . In this case we can work with strictly commutative (Adams graded) DGA’s. We approximate A by finite cell algebras where the cells are attached in negative Adams degrees. It is easily seen that such an algebra is of bounded Tate type (forgetting the differential it is free on finitely many generators in negative Adams degree).

The second way to approximate A (which we can assume to be of strict Tate type) is to truncate A in the following ways: For $i \in \mathbf{N}_{>0}$, $k < 0, n \in \mathbf{Z}$ we set $A_i(k)^n = 0$ if $n < ik$ and $A_i(k)^n = A(k)^n$ otherwise. Then the A_i form subalgebras of A such that $A = \operatorname{colim}_i A_i$, and each A_i is bounded.

This finishes the proof. \square

Remark 9.7. *The two strategies employed in the above proof to write a Tate-algebra as a filtered (homotopy) colimit of bounded Tate-algebras does not work for Tate algebras over the integers. For the first method we note that a finite cell algebra with generators in negative Adams degrees will in general not be bounded because of the contributions of the homology of the symmetric groups. The second method fails since the truncations considered will in general not be \mathbb{S} -algebras since the involved E_∞ -operad lives in negative cohomological degrees.*

10. EXAMPLES

Our general theorem (2.1) applies to all smooth schemes over fields with rational coefficients. In its form it deals with representation categories of pro affine derived group schemes. In this section we focus on examples of theorems (2.2) and (2.3) which discuss situations where Tate motives can be modelled as representation categories of affine derived group schemes.

We assume the reader is familiar with the constructions of affine derived group schemes starting with an Adams graded commutative \mathbb{S} -algebra in complexes with an augmentation. Below we will always apply theorems (6.21), (7.2), (7.4), remark (7.5) and theorems (9.5), (9.6).

10.1. First examples. Let k be a field of characteristic 0 and A the algebra in $\operatorname{Cpx}(k)^{\mathbf{Z}}$ where $A(0) = S^0(k)$, $A(-1) = S^{-1}(k \oplus k)$, and all other complexes are equal to 0. That is the typical cohomology algebra of $\mathbf{CP}^1 \setminus \{0, 1, \infty\}$ where we put the generators for H^1 in Adams degree -1 .

The algebra $B^1 = k \otimes_A^{\mathbf{L}} k$ sits completely in cohomological degree 0, and the Hopf algebra structure determines the pro-unipotent group scheme with \mathbb{G}_m -action which corresponds to the completed free Lie algebra over k on two generators in Adams degree 1.

If we put $A = k[x]/(x^{n+1})$, where x sits in cohomological degree 2 and Adams degree -1 , then A recovers the k -cohomology of \mathbf{CP}^n where we put the generator in cohomological degree 2 into Adams degree -1 . In this case the algebra $B^1 = k \otimes_A^{\mathbf{L}} k$ sits in various cohomological degrees.

In the case $A = k[x]$ with the bidegree of x as above we have that $B^1 = k \otimes_A^{\mathbf{L}} k$ has a single generator in cohomological degree 1 and Adams degree -1 .

In these cases our representation theorem applies and we have

$$\operatorname{Perf}(A) = \operatorname{Perf}(B^\bullet \rtimes \mathbb{G}_m).$$

Here $B^\bullet \rtimes \mathbb{G}_m$ is to be understood as in notation (9) from section (9).

Since these algebras are base changes from algebras over $\mathbf{1} \in \operatorname{Cpx}(\mathbf{Ab})^{\mathbf{Z}}$ our representation theorems apply to their versions over an arbitrary coefficient ring. In particular we have their versions with \mathbf{Z} -coefficients.

10.2. Finite fields. Next we give examples for base schemes and coefficients where our theorems (2.2) and (2.3) apply.

First let k be a finite field with $q = p^e$ elements. Let $R = \mathbf{Z}[\frac{1}{p}]$. Using the Bloch-Kato conjecture Levine computed the motivic cohomology of k ([11, Remark 14.11]):

$$\begin{aligned} \mathrm{Hom}_{\mathrm{DM}(k)_R}(R(0), R(0)) &= R, \\ \mathrm{Hom}_{\mathrm{DM}(k)_R}(R(0), R(n)[1]) &= R/(q^n - 1)R, \quad n \geq 1, \end{aligned}$$

and all other bi-homs are equal to 0. Thus theorem (2.3) applies and we have a derived motivic fundamental group B^\bullet over R such that $\mathrm{DMT}_{\mathrm{gm}}(k)_R \simeq \mathrm{Perf}(B^\bullet)$.

It is tempting to compute B^1 in this simple case. However it is unclear what the multiplication $A \otimes_R^{\mathbf{L}} A \rightarrow A$, A the algebra of bounded Tate type modelling the Tate motives, does on the Tor-terms.

10.3. Number fields. Next let k be a number field. Levine's computation ([11, Remark 14.11]) in this case gives an isomorphism

$$\mathrm{Hom}_{\mathrm{DM}(k)}(\mathbf{Z}_{(l)}(0), \mathbf{Z}_{(l)}(n)[i]) \otimes_{\mathbf{Z}_{(l)}} \mathbf{Z}_l \cong H_{\acute{e}t}^i(k, \mathbf{Z}_l(n))$$

for all $i \leq n$.

Thus theorem (2.2) applies and we have a derived motivic fundamental group B^\bullet over \mathbf{Z} such that

$$\mathrm{DMT}_{\mathrm{gm}}(k) \simeq \mathrm{Perf}(B^\bullet).$$

This fundamental group is the promised integral structure on the usual (non-derived) fundamental group for number fields.

10.4. Finite coefficients. We look now at the special case where the coefficients are an algebra over a finite field, so we suppose that R' is an $R = \mathbb{F}_p$ -algebra, p a prime. Suppose k is a field of characteristic not equal to p .

Then by the Bloch-Kato conjecture we have

$$\mathrm{Hom}_{\mathrm{DM}(k)_R}(R(0), R(n)[i]) \cong H_{\acute{e}t}^i(k, R(n)),$$

thus in particular the assumptions of theorem (2.3) are fulfilled. So we have derived motivic fundamental group B^\bullet over R' such that

$$\mathrm{DMT}_{\mathrm{gm}}(k)_{R'} \simeq \mathrm{Perf}(B^\bullet).$$

10.5. Geometric fundamental groups. Let R be an algebra fulfilling the assumptions of theorem (2.3). Suppose k gives rise to an algebra A over R of bounded Tate type, i.e. that the assumptions of theorem (2.3) are fulfilled as e.g. in all previous examples. Let $X = \mathbf{P}_k^1 \setminus \{0, 1, \infty\}$. We denote the corresponding algebra modelling Tate motives over X by $A(X)$. Let B^\bullet and $B^\bullet(X)$ be the derived affine schemes corresponding to A and $A(X)$. Then the natural map of cosimplicial algebras $B^\bullet \rightarrow B^\bullet(X)$ can be thought of as a map of group schemes in the opposite direction. The kernel of this group scheme map is an affine derived group scheme sitting completely in degree 0. It coincides as a group scheme with \mathbb{G}_m -action with our first example in this section defined over R . We think of this kernel as the geometric fundamental group of X .

In a similar way we can define geometric fundamental groups of other varieties over k such as \mathbf{P}_k^n or Grassmannians which admit stratifications where the strata are linear varieties.

10.6. Beilinson motives. Let \mathbf{LQ} be the Landweber spectrum over a base scheme S modelled on the rationals with its canonical E_∞ -structure, see [15, section 10]. Let $\mathrm{DMT}_{\mathrm{B},\mathrm{gm}}(S)$ be the full triangulated subcategory of the homotopy category of highly structured \mathbf{LQ} -modules spanned by the motivic spheres (note that the latter homotopy category is the category of Beilinson motives $\mathrm{DM}_{\mathrm{B}}(S)$ in the sense of [4]). Since \mathbf{LQ} is rational it is canonically strongly periodizable in the sense of [17]. Thus the representation theorem of loc. cit. applies and we find a commutative algebra A in $\mathrm{Cpx}(\mathbb{Q})^{\mathbf{Z}}$ such that $\mathrm{DMT}_{\mathrm{B},\mathrm{gm}}(S) \simeq \mathrm{Perf}(A)$. If we suppose that A is of bounded Tate type then we get an affine derived group scheme B^\bullet over \mathbb{Q} such that

$$\mathrm{DMT}_{\mathrm{B},\mathrm{gm}}(S) \simeq \mathrm{Perf}(B^\bullet).$$

In general, if A is of Tate type, we get a pro affine derived group scheme “ $\lim_i B_i^\bullet$ ” such that

$$\mathrm{DMT}_{\mathrm{B},\mathrm{gm}}(S) \simeq \mathrm{Perf}(\varprojlim_i B_i^\bullet).$$

The first case applies for example when the base is the spectrum of S -integers of a number field \mathcal{O}_S or $\mathbf{P}_{\mathcal{O}_S}^1 \setminus \{0, 1, \infty\}$.

In both cases the derived fundamental group sits completely in degree 0. In the case of the S -integers we recover the motivic fundamental group defined in [5]. The kernel of the induced map of fundamental groups is the geometric fundamental group defined in loc. cit. after forgetting the action of the motivic fundamental group of \mathcal{O}_S . To construct this action we would have to take care of a fiber functor induced by a (tangential) base point of $\mathbf{P}_{\mathcal{O}_S}^1 \setminus \{0, 1, \infty\}$ in our derived setting. We did not pursue that point further in this text.

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